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Bi-orthogonality relations for fluid-filled elastic cylindrical shells: Theory, generalisations and application to construct tailored Green's matrices

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ABSTRACT

The paper addresses the classical problem of time-harmonic forced vibrations of a fluid-filled cylindrical shell considered as a multi-modal waveguide carrying infinitely many waves. The forced vibration problem is solved using tailored Green's matrices formulated in terms of eigenfunction expansions. The formulation of Green's matrix is based on special (bi-)orthogonality relations between the eigenfunctions, which are derived here for the fluid-filled shell. Further, the relations are generalised to any multi-modal symmetric waveguide. Using the orthogonality relations the transcendental equation system is converted into algebraic modal equations that can be solved analytically. Upon formulation of Green's matrices the solution space is studied in terms of completeness and convergence (uniformity and rate). Special features and findings exposed only through this modal decomposition method are elaborated and the physical interpretation of the bi-orthogonality relation is discussed in relation to the total energy flow which leads to derivation of simplified equations for the energy flow components.

1. Introduction

In this paper we address the classical problem of time-harmonic wave propagation in a thin elastic fluid-filled cylindrical shell loaded by an inviscid compressible fluid without mean flow. This is a subject broadly covered in literature on applied mathematics, see e.g. Refs. [1–6]. Among other applications, this formulation is used to address transmission of vibro-acoustic energy which is of primary interest in e.g. the oil and gas industry as well as in larger pumping systems conveying waste water or distributing domestic water to inhabitants. While the analysis of free waves in such a waveguide is a well-established subject, a forced response in various excitation conditions has not yet been fully explored. To cover arbitrarily distributed acoustic and structural sources it is convenient to derive Green's matrices i.e. to study the response to an excitation modelled as delta-functions. In this formulation of the problem it is expedient, on the one hand, to consider detailed analysis of the energy redistribution and mode conversion in the near-field to gain additional physical insight. On the other hand, the mathematical issues of completeness and convergence need to be addressed.

To understand the energy redistribution and mode conversion in the near-field e.g. from pump to pipe or across flange connections, an accurate coupled vibro-acoustic model of an infinite pipe needs to be formulated. In this paper we adopt the tailored Green's function/matrices as introduced in Ref. [5]. These functions deviate from the canonical free-space Green's function of acoustics, in that they satisfy additional boundary conditions – continuity at the fluid-structure interface. Here we consider only

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Nomenclature

c_{fl}	Fluid sound speed - [m s^{-1}]
c_{str}	Structural sound speed $\equiv \sqrt{\frac{E}{\rho_{str}(1-\nu^2)}}$ - [m s^{-1}]
E	Young's modulus - [Pa]
h	Shell thickness - [m]
k	Non-dim. axial wave-number $\equiv \widetilde{k}R$
m	Non-dim. circumferential wave-number
N^u	Non-dim. axial energy flow $\equiv \frac{1-\nu^2}{EhRc_{str}}\widetilde{N}^u$
N^v	Non-dim. torsion energy flow $\equiv \frac{1-\nu^2}{EhRc_{str}}\widetilde{N}^v$
N^w	Non-dim. transverse energy flow $\equiv \frac{1-\nu^2}{EhRc_{str}}\widetilde{N}^w$
$N^{w'}$	Non-dim. bending energy flow $\equiv \frac{1-\nu^2}{EhRc_{str}}\widetilde{N}^{w'}$
N^ϑ	Non-dim. acoustic energy flow $\equiv \frac{1-\nu^2}{EhRc_{str}}\widetilde{N}^\vartheta$
N^Σ	Non-dim. total energy flow $\equiv \frac{1-\nu^2}{EhRc_{str}}\widetilde{N}^\Sigma$
p	Non-dim. acoustic pressure $\equiv \frac{1}{c_{fl}^2\rho_{fl}}\widetilde{p}$
P	Non-dim. amplitude of acoustic pressure
q_l	Non-dim. external structural forces $\equiv \frac{1-\nu^2}{E}\widetilde{q}_l$ $(l = 1, 2, 3)$
Q_4	Non-dim. moment $\equiv \frac{1-\nu^2}{Eh^2}\widetilde{Q}_4$
Q_l	Non-dim. forces $\equiv \frac{1-\nu^2}{Eh}\widetilde{Q}_l$ $(l = 1, 2, 3)$
$\mathcal{Q}_l$	Non-dim. amplitude of structural forces/-moment $(l = 1, \ldots, 4)$
r	Non-dim. radial coordinate $\equiv \frac{\widetilde{r}}{R}$
r_0	Excitation point in r
R	Shell radius - [m]
T	Non-dim. external acoustic source $\equiv \frac{R}{c_{fl}}\widetilde{T}$
u	Non-dim. axial displacement $\equiv \frac{\widetilde{u}}{R}$
U	Non-dim. axial amplitude
v	Non-dim. circumferential displacement $\equiv \frac{\widetilde{v}}{R}$
V	Non-dim. circumferential amplitude
w	Non-dim. radial displacement $\equiv \frac{\widetilde{w}}{R}$
W	Non-dim. radial amplitude
w'	Non-dim. rotation
W'	Non-dim. rotation amplitude
x	Non-dim. axial coordinate $\equiv \frac{\widetilde{x}}{R}$
γ	Non-dim. sound speed ratio $\equiv \frac{c_{str}}{c_{fl}}$
ϑ	Non-dim. acoustic velocity $\equiv \frac{\widetilde{\vartheta}}{c_{fl}}$
$\mathcal{V}$	Non-dim. amplitude of acoustic velocity
θ	Non-dim. circumferential coordinate
κ	Non-dim. radial wave-number $\equiv \sqrt{k^2 + \gamma^2\Omega^2}$
μ	Non-dim. thickness-to-radius ratio $\equiv \frac{h}{R}$
ν	Poisson's ratio - [−]
ξ	Excitation point in x
ρ_{fl}	Fluid density - [kg m^{-3}]
ρ_{str}	Structural density - [kg m^{-3}]
ρ	Non-dim. density ratio $\equiv \frac{\rho_{fl}}{\rho_{str}}$
ϕ	Non-dim. velocity potential $\equiv \frac{\widetilde{\phi}}{c_{fl}R}$
Φ	Non-dim. amplitude of velocity potential
ω	Angular frequency - [rad s^{-1}]
Ω	Non-dim. frequency $\equiv \frac{\omega R}{c_{str}}$
$J_m(x)$	Bessel-function of first kind of order $m \in \mathbb{Z}$
$\delta(x)$	Dirac delta-function
$\mathrm{sgn}(x)$	Signum function
$\lvert x\rvert$	Module of x
i	Complex operator
$*$	Complex conjugated
$'$	Derivative with respect to x
$\mathbf{U}$	U as a matrix or vector
$\bar{}$	Indicates modal coefficients
$\sim$	Indicates dimensional quantities
$0F$	Indicates loading condition
$\overset{(n)}{m}$	Modal components of circumferential, m, and axial wave-number, n, e.g. $\left(k_m^{(n)} \in \mathbb{C} \mid n, m \in \mathbb{Z}, n \neq 0\right)$

the tailored Green's matrices (excitation by ideal sources), while the generation of vibro-acoustic energy internally in a pump is not treated here. Due to the versatility of Green's formulation, see e.g. Refs. [7,8], we can easily generalise to arbitrary sources generated by a pump or to finite and/or compound pipes with arbitrary boundary conditions and/or transition properties using the Boundary Integral Equations Method (BIEM), see e.g. Refs. [2–4,7–14]. However, in the heavy fluid-loading format the problem becomes transcendental and the accuracy of the near-field solution is compromised by the computational efficiency when solved using the conventional weak solution form (integral average). Thus, the purpose of this paper is to improve both accuracy and computational efficiency of the solution by solving the forced vibration problem using modal decomposition (strong form).

The formulation of Green's matrix is based on the eigenfunction expansion method with the eigenvalues derived from the dispersion equation. This method is the most commonly used method in vibro-acoustic problems. In Refs. [3,15–17] authors have employed specially derived orthogonality relations to decompose the governing equations into uncoupled algebraic modal equations which can easily be solved analytically; providing the strong solution form of Green's matrix. The decomposition is analogue to the decomposition of circumferential modes by orthogonality of trigonometric functions, see e.g. Refs. [2–4,13,14], however, with more advanced orthogonality relations between the involved eigenfunctions. In Ref. [3] this modal decomposition method was used for the acoustic duct where the 'more advanced' orthogonality relation reduces to orthogonality of cylindrical functions i.e. Bessel-functions, see relation in e.g. Refs. [18–20]. On the other hand, similar relations have been derived in Refs. [16,17,21–28] for plates, strips, layers, laminates, springs, beams, shells etc. and facilitated in e.g. Refs. [15–17] to analytically