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On application of the Floquet theory for radially periodic membranes and plates

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ABSTRACT

The paper is concerned with the vibro-isolation effects in radially periodic membranes and plates. Alternative formulations of the canonical Floquet theory for analysis of wave propagation in these elastic structures are compared with each other. An extension of this theory beyond the applicability limits of the well-known theory of Bragg fiber is proposed. The similarities and differences in performance of infinite and finite structures periodic in Cartesian and polar coordinates are highlighted and explained.

1. Introduction

Analysis of wave propagation in periodic structures is a classical topic in theory of guided waves, which has been studied for decades in various realms of physics. Acoustic and electromagnetic (in particular, optical) waveguides have gained much attention from this viewpoint, although periodicity effects have also been studied in the theory of elastodynamics, the thin plate/shell theory and in geophysics. The periodicity effects are well-known and understood for a waveguide in Cartesian coordinates regardless its dimension (in terms of structural mechanics, a rod/beam, a plate or a three-dimensional lattice). The underlying theory customarily referred to as the Floquet theory (or the Floquet theorem) relies on the translational invariance of the problem formulation for an infinite periodic structure. References [1–6] are just a few of the canonical and the recent publications on this subject concerned with the issues of vibro-isolation. To the date, it may be concluded that further advances in analysis of the wave propagation in structures, periodic in Cartesian coordinates, should be related to complicating effects such as uncertainties, transients, non-linearity, and similar [5–8].

In practice of vibro-acoustics, it is also of interest to consider periodic structures in other coordinate systems. Specifically, a compact source of vibrations (e.g., an operating pump) may be installed at a relatively large and flexible plate (e.g., a ship's deck). To avoid transmission of the structure-borne sound, a sequence of co-centric circular strips with alternating properties may be deployed with the point, where a pump is mounted, as the centre. It is natural to use the polar coordinates to describe the wave propagation in such a structure. The analysis of cylindrical elastic waves is the canonical subject fully covered in many references with the monographs [9–11] being just a few examples. One may expect that the destructive interference of incident and reflected waves in a radially periodic structure produces the stop band effect as is known for a chessboard-type one.

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Obviously, in the far-field zone, this performance must asymptotically match the predictions obtained in Cartesian coordinates.

A problem of wave propagation in a radially periodic waveguide has been formulated long time ago with the reference to the guiding of light and its approximate solution is well-known in optics as the theory of Bragg fiber [12]. The challenges of 'adjusting' the Bloch theorem for a cylindrically symmetric Bragg fiber have been acknowledged in Refs. [13–16]. Alternative "tuning" techniques are proposed in these references, with the simplest one being replacement of Bessel functions with their far-field approximations. In Ref. [15], the translational invariance is recovered by special continuous variations of density and bulk modulus of an anisotropic fluid. In Ref. [16], an approximation in formulation of Bessel equation is used and it is claimed to be more accurate, than the approach used in Refs. [12,13].

Recent papers [17–20] address numerical analysis of phononic band structures under assumption that the distance from the origin of polar coordinates is large enough to accept the translational symmetry of a radial phononic crystal. In Refs. [17–19], the standard equations of elasto-dynamics are numerically solved in cylindrical coordinates by means of commercially available software. In Ref. [20], such a numerical analysis is performed both within the thin plate theory and the elasto-dynamics.

The applicability of "straightforward" periodicity conditions (the theory of Bragg fiber) in the near-field, i.e., in the relative vicinity of the origin of the polar coordinates needs closer inspection, and this constitutes the main research goal of the article. It should be noticed that, from the mathematically rigorous viewpoint, it is a desperate undertaking to tune, or adjust the Floquet theory to deal with the propagation of cylindrical elastic waves in structures periodic in polar coordinates, see Refs. [12–16]. From the practical viewpoint, however, it is plausible to explore possibilities of vibro-isolation in such a case, and this has been the motivation for the authors to revisit the theory of Bragg fibers in the framework of simple models of structural dynamics.

The paper is concerned with two case-studies. The first one is the wave propagation in a radially periodic elastic membrane exposed to the uniform tension. In Section 2, the location of pass and stop bands in this structure is compared with the power flow analysis for an infinite membrane with a periodic insert and with the distribution of eigenfrequencies of finite structures composed of several periodicity cells. The same analysis for a radially periodic plate constitutes the second case-study presented in Section 3.

2. A radially periodic membrane

In the one-dimensional case, the Helmholtz equation, which describes time-harmonic plane wave propagation in an acoustic duct, or in an elastic string, or in an elastic rod, is the simplest model to formulate the Floquet theory for periodic structure composed of continuous constituents. This equation converted to polar coordinates describes time harmonic propagation of the cylindrical wave of dilatation in a membrane under constant uniform tension, and is perfectly suited for analysis of periodicity effects in this system of coordinates.

2.1. The energy flow in an infinite periodic membrane with finite number of periodicity cells

We consider a radially periodic membrane shown on Fig. 1:

The "black" and "white" segments have different densities and lengths, but are exposed to the same uniform tension. The length of the "white" segment is taken as a scale to measure the radial coordinate, see Fig. 2. The governing equation for each segment is:

$$u_i''(r) + \frac{1}{r}u_i'(r) + \left(k_i^2 - \frac{m^2}{r^2}\right)u_i(r) = -q(r) \tag{1}$$

In Eq. (1) m is the circumferential wavenumber, $u_i(r)$ and $q_i(r)$ are the transverse displacement and external distributed load at the given wavenumber m in the i-th segment of a membrane. The radial wavenumber is designated as $k_i = \frac{\omega}{c_i}$, $c_i = \sqrt{\frac{T_0}{\rho_i}}$, where $T_0 = const$ - membrane tension.

The following dimensionless parameters are used in this work:

$$\gamma = \frac{l_2}{l_1};\ \ \sigma = \frac{c_2}{c_1};\ \ k_1 l_1 = \Omega \tag{2}$$

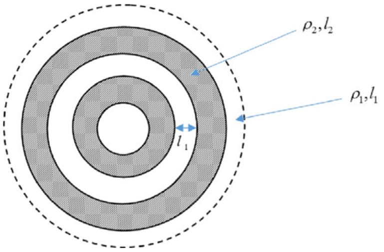

Fig. 1. A radially periodic elastic membrane under uniform tension.