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# The effect of generalised force correlations on the response statistics of a harmonically driven random system

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ABSTRACT

If the physical properties of a structural component are sufficiently random then the statistical distribution of the natural frequencies and mode shapes tends to a universal distribution associated with the Gaussian Orthogonal Ensemble (GOE) of random matrices. Previous work has exploited this result to yield expressions for the relative variance of the energy of the response of a random system to harmonic excitation. The derivation of these expressions employed random point process theory, and in the theoretical development it was assumed that the modal generalised forces were uncorrelated. Although this assumption is often valid, there are cases in which correlations between the generalised forces can significantly affect the response variance, and in the present work the existing theory is extended to include correlations of this type. The extended theory is applicable to both single frequency responses and to band average responses, and the developed closed form expressions are validated by comparison with direct simulations for a random plate structure.



## 1. Introduction

The vibrational response of an engineering system can be very sensitive to imperfections, and there is much practical interest in being able to predict not just the response of a nominally perfect structure but also the statistical spread of the response arising from uncertainties. This can be an extremely difficult task, since a computational model of a car or an aeroplane may contain millions of degrees of freedom, and the statistics of the uncertainties in the properties of the system may not be known. In many cases however the system can be sufficiently random for certain universal statistical laws to apply to the distribution of the natural frequencies and the mode shapes and this means that, almost counterintuitively, the statistics of the response can be predicted without any detailed knowledge of the uncertainties in the system. Early work in this area was performed by Lyon [1], who derived formulae for the mean and variance of the power input to a system subjected to single point harmonic forcing on the assumption that the natural frequencies constitute a Poisson random point process. The effect of repulsion between natural frequencies (not included in the Poisson model) was also considered in reference [1], and this work was extended by Davy [2] to the case of multiple excitation and response points. It was later recognised that in many practical cases the natural frequencies of a random system (without symmetries) do not have the

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distribution assumed in References [1,2]; more commonly, the natural frequencies have the same distribution as the eigenvalues of a random matrix ensemble known as the Gaussian Orthogonal Ensemble (GOE) [3], as demonstrated experimentally for an aluminium block by Weaver [4]. The applicability of this distribution is reviewed in reference [5], and the occurrence of the distribution has been exploited to predict of the response statistics of single structural components (for example [6–8]) and acoustic volumes (for example [9,10]). Work of this type has also been extended to the prediction of the response of built-up systems within the context of the Statistical Energy Analysis (SEA) approach to structural dynamics [11]. As described below, the present work provides an enhancement of previous work on the application of the GOE to vibrating systems by relaxing one of the assumptions employed in the analysis, thereby allowing the approach to be applied to a wider range of problems. Of course, the literature regarding the response of random dynamic systems is vast and there are many approaches that can be employed to predict the statistics of the response of such a system. Because the present work is narrowly focussed on extending the range of an existing method, the previous work referenced here is exclusively focussed on material directly related to the earlier results. Wider views on the treatment of uncertainties in computational models can be found in texts such as [12].

The usual approach to the analysis of the dynamic response of a linear structural component is to represent the response as a sum of modal contributions; each modal contribution involves the frequency response function of the mode, the generalised force in the mode, and the mode shape. If the system natural frequencies are taken to be random, then the modal sum is exactly the type of quantity that is amenable to analysis by random point process theory [8–10]. In this theory the response is viewed as a sum of frequency response functions that have random amplitudes, and given the distribution of the natural frequencies (arising from the GOE for example), the theory yields expressions for the statistical moments of the response. This approach was used in References [7,8] to yield closed form expressions for the mean and the variance of the energy of the response, and the results were compared favourably with both detailed simulations and experimental measurements. Reference [7] was concerned with the response at a single forcing frequency, whereas reference [8] extended this result to the statistics of a band-averaged response. A key feature of random point process theory, as presented in References [13–15] and exploited in References [7,8], is that the random amplitudes in the modal sum are taken to be statistically independent and identically distributed. Although this is a perfectly reasonable assumption in many cases, it has been found in the course of the present work that significant errors can arise if the system is subjected to random loading and the modal overlap is relatively high. It has also been found that the key failing in the existing theory is the neglect of correlations between the random amplitudes, arising from correlations between the generalised forces, and hence the objective of the present work is to extend the current theory for single frequency and band-averaged responses by including the effect of such correlations. The theory is extended in a general way, so that the correlation between the random amplitudes may have any cause, although particular attention is paid to the example of correlations arising from random forces. To add a practical context to the work, it can be noted that harmonic loading with a random or uncertain amplitude can arise (for example) from rotating or reciprocating machinery. The frequency of operation may be well defined, but the loading generated by imbalance or internal mechanisms may vary randomly across different operating conditions and across different machines from the same production line.

In Section 2 below, the existing theory for predicting the response variance of a structural component is summarised, and an example is given for which the theory provides poor results. Random point process theory is then extended in Section 3 to include the effect of correlations between terms in the modal sum, and a new closed form expression is obtained for the relative variance under harmonic excitation. The analysis is extended in Section 4 to the relative variance of the band-averaged response, and in Section 5 both new results are compared with numerical simulations of a random plate. The findings of the work are summarised in Section 6.

## 2. Preliminary considerations

The response of a proportionally damped linear system to harmonic forcing at frequency $\omega$ can be written as a modal sum in the form

$$u(\omega, \mathbf{x}) = \sum_n \frac{i\omega g_n \phi_n(\mathbf{x})}{\omega_n^2 - \omega^2 + i\eta\omega_n^2}, \tag{1}$$

$$g_n = \int_R P(\mathbf{x})\phi_n(\mathbf{x})\mathrm{d}\mathbf{x}, \tag{2}$$

where $u(\omega, \mathbf{x})$ is the complex amplitude of the velocity at spatial point $\mathbf{x}$, $\omega_n$ and $\phi_n(\mathbf{x})$ are respectively the natural frequency and the mode shape of the $n$th mode of vibration, $\eta$ is the loss factor, $P(\mathbf{x})$ is a distributed load, and $g_n$ is the resulting generalised force in the $n$th mode. For simplicity both the velocity and the mode shape have been taken to be scalar quantities in Eq. (1), but both could readily be generalised to vector quantities. The time averaged kinetic energy of the system is given by

$$T(\omega) = (1/4)\int_R \rho(\mathbf{x})|u(\omega, \mathbf{x})|^2 \mathrm{d}\mathbf{x}, \tag{3}$$