



Exact steady states of the periodically forced and damped Duffing oscillator



Alexander F. Vakakis^a, Antoine Blanchard^{b,*}

^a Department of Mechanical Science and Engineering, University of Illinois at Urbana–Champaign, Urbana, IL 61801, USA

^b Department of Aerospace Engineering, University of Illinois at Urbana–Champaign, Urbana, IL 61801, USA

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ABSTRACT

We present an analytical methodology to compute the exact steady states of the strongly nonlinear Duffing oscillator with arbitrary viscous damping and under the action of a time-periodic excitation. In contrast to current analytical approaches that consider harmonic excitations to produce approximate steady-state solutions, the excitations considered in this work are multi-harmonic and are computed in closed form. We consider the exact fundamental resonance plot of this strongly nonlinear system and present an exact depiction of the dynamic balance between the internal and external forces leading to it in terms of rotating vectors in the complex plane with modulated amplitudes and modulated frequencies of rotation. Moreover, an infinity of exact steady-state solutions seems to be possible, each corresponding to a different time-periodic excitation. Generalization of the presented methodology to systems with other type of nonlinearities or with many degrees of freedom is presumably straightforward.

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1. Introduction

The nonlinear Duffing oscillator with linear viscous damping and constant coefficients

$$\ddot{x}(t) + \alpha x(t) + \mu x^3(t) + \lambda \dot{x}(t) = F(t)$$

has been studied extensively in the literature due to its many applications in physics, applied mathematics and engineering. It is well known that the unforced and undamped system (with $F(t) = 0$ and $\lambda = 0$) is exactly solvable by quadratures given any arbitrary initial conditions [1]. The unforced and damped system, however, does not admit an exact analytical solution in terms of known tabulated functions [1,2], although approximate solutions can be developed using asymptotic methods, e.g., the method of multiple scales [3].

The Duffing oscillator with harmonic forcing has received considerable attention, with emphasis laid on its steady-state solutions. This system also does not admit exact solutions; however, under the assumption of weak nonlinearity and damping, asymptotic methods have been applied to study various interesting steady-state responses, including fundamental, subharmonic and superharmonic resonances [3], as well as chaotic responses (homoclinic tangles and Smale horseshoe chaotic maps) for special values of the parameters [4]. Indeed, harmonic excitation is a common assumption in studies focusing on the steady-state responses of the forced Duffing oscillator. This is likely due to the fact that for weak nonlinearity, the use of harmonic forcing functions enables the application of asymptotic techniques, and the reduction of the problem to a hierarchy of subprob-

* Corresponding author.

E-mail addresses: avakakis@illinois.edu (A.F. Vakakis), ablancha@illinois.edu (A. Blanchard).

lems (at different orders of approximation) with linear (i.e., harmonic) generating solutions. A notable exception is the body of works by Pilipchuk [5,6], who developed an analytical technique based on non-smooth generating solutions, and which is valid in strongly nonlinear response regimes. Even in that case, however, the analysis of the steady-state response involves the use of harmonic forcing functions [6] or other special forcing functions (e.g., pulse trains [7]), and the results are only approximate.

Hence, reconsidering the forced Duffing oscillator with strong nonlinearity and arbitrary viscous damping, *it is natural to ask if a harmonic forcing function is the most appropriate excitation for producing an exact steady-state response in this system*. For a linear oscillator, this indeed is the case, but *given the strong nonlinearity of the Duffing oscillator, one may wonder if a different time-periodic, yet not necessarily harmonic, function is required to produce an exact steady-state solution*. This basic question is the main topic of this work. We mention at this point that the concept of “exact steady state” of a strongly nonlinear discrete system was first introduced by Rosenberg [8,9] and Hsu [10] (see also [11]) assuming, however, no damping. In that case, for an appropriate periodic excitation, the exact steady state was reduced to an equivalent undamped free oscillation, which could be solved exactly by quadratures. This technique was subsequently used to determine the class of periodic functions that produce exact steady-state solutions in undamped coupled oscillators [12,13]. Two recent works by Kovacic [14,15] extend these results to computing the exact steady states of one-degree-of-freedom and coupled nonlinear oscillators. Yet, the problem of computing exact steady-state solutions of forced and damped discrete or continuous oscillators remains open. This work aims to initiate the study of this problem by providing a constructive methodology for computing the exact solution of the Duffing oscillator for arbitrary damping and appropriately defined periodic forcing.

2. Exact nonlinear steady states

Before we consider the exact resonance of the forced and damped Duffing oscillator, we make a digression which also serves as a motivational example. In particular, we consider the fundamental resonance of the single-degree-of-freedom *linear* damped oscillator subject to periodic forcing. Although this is a classic and well-studied problem, we wish to pose it from a new perspective by asking the following basic question: *What type of periodic excitation leads to an exact fundamental resonance in this system?* For the linear case, the answer to this question is well known, i.e., the applied force should be harmonic and the resulting steady-state response is also harmonic, albeit with a phase shift with respect to the excitation. However, we wish to reformulate this classic result in the context of the previously posed question. To this end, we consider the normalized governing equation of motion

$$\ddot{x}(t) + x(t) + \lambda \dot{x}(t) = F(t), \quad \lambda > 0, \quad (1)$$

where $F(t)$ is the applied periodic force that is yet to be determined. Assuming that a fundamental resonance has been reached, (i.e., a steady-state solution where the response and the excitation have the same frequency), we decompose the excitation in terms of “undamped” and “damped” components ($F_u(t)$ and $F_d(t)$, respectively), and express it in the form $F(t) = F_u(t) + F_d(t)$. In what follows, we will compute these two force components separately so that they satisfy the “undamped” and “damped” subproblems given by

$$\ddot{x}(t) + x(t) = F_u(t), \quad (2a)$$

$$\lambda \dot{x}(t) = F_d(t), \quad (2b)$$

respectively.

Considering first the undamped subproblem (2a), and recognizing that the periodic excitation and the steady-state response should have the same frequency, we require that the undamped force component $F_u(t)$ be proportional to the steady-state response $x(t)$, i.e., $F_u(t) = -Ux(t)$, where $U \in \mathbb{R}$ is a multiplicative factor. Then, Eq. (2a) yields

$$\ddot{x}(t) + (1 + U)x(t) = 0 \quad \Rightarrow \quad x(t) = A \cos \omega t, \quad U = \omega^2 - 1. \quad (3)$$

Relation (3) computes the exact steady-state response, with A denoting the amplitude and ω the frequency of the steady-state response. We note that in Eq. (3), the multiplicative factor U is directly related to the frequency of the steady-state oscillation. Once the solution (3) is substituted into the damped subproblem (2b), the damped force component $F_d(t)$ required for realizing the exact fundamental resonance is fully determined. Following this procedure, we recover the well-known result that, for fundamental resonance of Eq. (1), the periodic force should be computed as

$$F(t) = \underbrace{(1 - \omega^2)A \cos \omega t}_{F_u(t)} \underbrace{- \lambda A \omega \sin \omega t}_{F_d(t)} = A \sqrt{(1 - \omega^2)^2 + (\lambda \omega)^2} \cos(\omega t - \phi), \quad (4a)$$

$$\tan \phi = \lambda \omega / (\omega^2 - 1), \quad (4b)$$

where ϕ denotes the phase difference between the excitation and the response. Moreover, if we denote by P the magnitude of the applied excitation, we obtain the compact expression $F(t) = P \cos(\omega t - \phi)$, and the well-known *linear resonance expression*

$$P = A \sqrt{(1 - \omega^2)^2 + (\lambda \omega)^2}, \quad (5)$$

which relates the magnitude of the applied excitation P to the amplitude A and frequency ω of the steady-state response for a given value of the viscous damping λ . This completes the derivation for the linear problem, and proves that the only periodic

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