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The influence of phase-locking on internal resonance from a nonlinear normal mode perspective

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ABSTRACT

When a nonlinear system is expressed in terms of the modes of the equivalent linear system, the nonlinearity often leads to modal coupling terms between the linear modes. In this paper it is shown that, for a system to exhibit an internal resonance between modes, a particular type of nonlinear coupling term is required. Such terms impose a phase condition between linear modes, and hence are denoted *phase-locking* terms. The effect of additional modes that are not coupled via phase-locking terms is then investigated by considering the backbone curves of the system. Using the example of a two-mode model of a taut horizontal cable, the backbone curves are derived for both the case where phase-locked coupling terms exist, and where there are no phase-locked coupling terms. Following this, an analytical method for determining stability is used to show that phase-locking terms are required for internal resonance to occur. Finally, the effect of non-phase-locked modes is investigated and it is shown that they lead to a *stiffening* of the system. Using the cable example, a physical interpretation of this is provided.

1. Introduction

Weakly nonlinear systems typically have an underlying linear structure and the underlying linear modes are often used to describe the fundamental components of the system. In this paper we investigate the coupling terms that are present in nonlinear systems when projected onto these linear modes. Specifically, we show how phase-locking conditions in these terms influence the internally resonant dynamic behaviour. Such phenomena are often observed in forced, lightly damped, and weakly nonlinear systems with multiple degrees-of-freedom, which represent a variety of physical applications, see for example [1–4]. Here, internal resonance is defined as the triggering of a dynamic response of a linear mode of the system that is not subjected to direct external excitation. Many previous authors have considered problems of this type, see for example [3,4] and references therein.

In this work we will use the normal form technique proposed by [5] for multi-degree-of-freedom forced, damped, weakly nonlinear systems. This approach leads naturally to the analysis of backbone curves, which define the dynamic behaviour of periodic motions in the unforced and undamped equivalent system, in the amplitude vs frequency plane. These are equivalent to the loci of the nonlinear normal modes (NNMs), represented in the amplitude vs frequency projection.

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Many authors have considered the NNMs of, for example, a two-degree-of-freedom spring–mass system [6–8]. Lewandowski [9] pointed out that bifurcations can occur in the backbone curves of this type, and the same author went on to analyse beam, membrane and plate examples [10]. More recently, the current authors have used backbone curves to study internal resonance phenomena in systems of coupled nonlinear oscillators [11–14]. In particular, bifurcations of backbone curves were used to indicate where an internal resonance may by triggered, however additional solution branches have also been observed that do not trigger internal resonances. As internal resonance is defined here as a behaviour seen in systems subject to external excitation, and as the backbone curves describe the unforced and undamped responses, it should be noted that the backbone curves themselves do not exhibit internal resonance. However, backbone curves do uncover modal interactions which may lead to internal resonances when external forcing and damping are introduced.

Here, the phenomenon of phase-locking between modes during internal resonance is considered in detail. Much of this discussion is motivated by the example of a taut cable, introduced in Section 2. Expressions for the backbone curves of the cable are developed in Section 3 by considering the interactions between pairs of linear modes. This analysis reveals that, depending on the pairs of modes that are considered, the backbone curves are either phase-locked or phase-unlocked.

One significant feature of phase-locking is revealed in Section 4, where it is shown that phase-locked terms are required for internal resonance. This is demonstrated using an analytical stability analysis, which considers the stability of the zero-amplitude solution of an unforced linear mode of a general weakly nonlinear system. This general analysis is applied to the cable example, demonstrating the physical significance of this observation. Lastly, in Section 5, the cable example is used to investigate the influence of additional, phase-unlocked, modes on the dynamic behaviour of an internally resonant pair of modes. It is shown that, although this type of mode cannot lead to additional internal resonances, they can impose a stiffening effect on the system, altering the response of the phase-locked pair. For the cable, this effect can be explained physically as an increase in the axial tension in the cable, due to the presence of additional modal oscillations.

2. Resonant equations of motion

Weakly nonlinear, multi-degree-of-freedom systems are often expressed in terms of the modal coordinates for the linearised version of the system, as the linear terms will then be decoupled. However, decoupling of the nonlinear terms is typically not achieved via a linear modal transform, and hence the modes will not, in general, match the NNMs of the nonlinear system. Note that here we use the term *modes* to refer to the modes of the linearised system. A multimodal nonlinear system may be written in modal coordinates, $\mathbf{q}$, as

$$\ddot{\mathbf{q}}+\Gamma\dot{\mathbf{q}}+\Lambda\mathbf{q}+\mathbf{N}(\mathbf{q})=\mathbf{f}. \tag{1}$$

Assuming linear modal damping, the kth diagonal elements in diagonal matrices Γ and Λ are $2\zeta_k\omega_{nk}$ and ω_{nk}^2 respectively and the vector $\mathbf{N}$ contains the nonlinear stiffness terms, and $\mathbf{f}$ represents the external excitation vector in modal coordinates. Here, ζ_k and ω_{nk} are used to denote the linear damping ratio and linear natural frequency of the kth mode respectively.

To analyse weakly nonlinear systems it is helpful to transform the equations of motion into a new set of coordinates, $\mathbf{u}$, which describe only the resonant components of the response. The dynamic equation in $\mathbf{u}$ is termed the *resonant* equation of motion. This can then be used to find steady-state solutions, in terms of modal amplitudes, via an exact harmonic balance using trial solutions of the form

$$u_k=U_k\cos(\omega_{rk}t-\phi_k)=\frac{U_k}{2}\mathrm{e}^{\mathbf{j}(\omega_{rk}t-\phi_k)}+\frac{U_k}{2}\mathrm{e}^{-\mathbf{j}(\omega_{rk}t-\phi_k)}=u_{pk}+u_{mk}, \tag{2}$$

where ω_{rk} is the response frequency of the kth mode and subscripts p and m indicate positive and negative (minus) complex exponential terms respectively. The introduction of ω_{rk} allows for the detuning of the kth mode from the linear natural frequency, ω_{nk}. For a resonant response, this response frequency is typically selected such that it is close to the linear natural frequency of the mode in question, $\omega_{rk}\approx\omega_{nk}$. Note, however, that response frequencies that are not close to the linear natural frequencies may also be selected, as the assumption that this detuning is small is not a requirement of the technique. Normal form analysis allows us to find the periodic responses of a system, as is assumed when computing NNMs, or the steady-state response to a sinusoidal forcing. Taking the period to be $T=2\pi/\Omega$, the response frequency of the kth mode, ω_{rk}, is an integer multiple of Ω. Likewise, if the system is forced at a single frequency, the forcing frequency is an integer multiple of Ω.

If forcing is near-resonant, i.e. the frequency of the forcing acting on any mode is close to the natural frequency of that mode, or if there is no forcing, then the resonant equation of motion is found by applying a nonlinear near-identity transform to Eq. (1) to give

$$\ddot{\mathbf{q}}+\Gamma\dot{\mathbf{q}}+\Lambda\mathbf{q}+\mathbf{N}(\mathbf{q})=\mathbf{f}\xrightarrow{\mathbf{q}=\mathbf{u}+\mathbf{h}(\mathbf{u})}\ddot{\mathbf{u}}+\Gamma\dot{\mathbf{u}}+\Lambda\mathbf{u}+\mathbf{N}_u(\mathbf{u})=\mathbf{f}, \tag{3}$$

where $\mathbf{h}$ is a vector of harmonic components. In the formulation shown in Eq. (3), it is assumed that the forcing and damping terms are resonant, and so are retained in the equation for $\mathbf{u}$. For further discussion of how non-resonant terms are handled see [5,15]. Additionally, details of how the harmonics $\mathbf{h}$ and the transformed nonlinear terms $\mathbf{N}_u$ are found are given in the Appendix.