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Equilibria and vibration of a buckled beam with attached masses or spring–mass systems

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ABSTRACT

A buckled beam with immovable pinned ends is considered. Attached to the beam are either one concentrated mass, two concentrated masses, a spring–mass system (that could model a human, robot, or passive vibration absorber), or a horizontal rigid bar with two vertical end springs (a “bounce–pitch” system that could model an animal or a vehicle). In the theoretical analysis, the beam is modeled as an inextensible elastica. Equilibrium configurations are determined first. Then small free vibrations about equilibrium are examined, and the lowest frequencies and corresponding modes are computed. The effects of various parameters are investigated, such as the ratio of the span to the total arc length of the beam, the locations and weights of the attached masses and systems, and the stiffnesses of the springs. For the case of a single attached mass, experiments are conducted and the results are compared to the theoretical ones.

1. Introduction

This study was motivated by a video of goats frolicking on a wide buckled beam in a pasture [1]. The goats are modeled here as point masses, or a mass with a spring representing the legs, or a rigid bar with springs. An analysis is developed, along with experiments on a buckled beam with an attached mass. Equilibrium shapes and vibration modes and frequencies are determined. The results may be applicable to curved systems (e.g., buckled beams, arches, and cylindrical panels) supporting equipment, vehicles, or vibration absorbers.

Papers that have investigated the dynamic behavior of buckled beams with attached masses include [2–7]. In most of these studies, the beam is subjected to lateral harmonic motion, and dynamic snap-through is examined. A spring–mass system is sometimes utilized to model the gait of humans or robots [8,9]; its application as a dynamic vibration absorber on a curved beam/panel is discussed in [10]. The “bounce–pitch” model of a horizontal rigid bar with vertical springs at its ends has been used to represent a vehicle [11].

The basic analysis will be described in Section 2 and the experiments in Section 3. Numerical results will be presented in Sections 4–7, respectively, for one attached concentrated mass, two masses, a spring–mass system, and a bounce–pitch system. Finally, concluding remarks will be given in Section 8.

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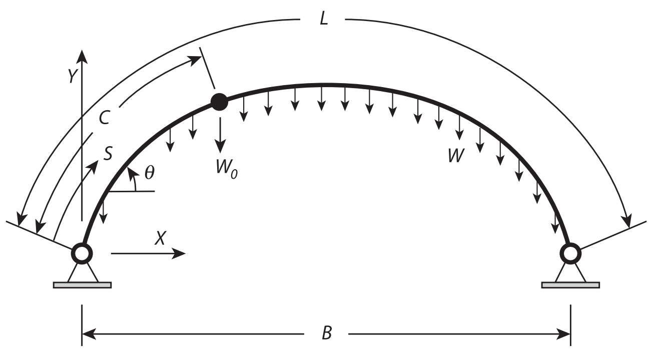

Fig. 1. Geometry of buckled beam with attached weight.

2. Formulation

A uniform, wide, linearly elastic beam is considered, modeled as an inextensible elastica. It is unstrained when straight. The total arc length is L, the modulus of elasticity is E, the moment of inertia of the cross section is I, Poisson's ratio is ν, and the weight per unit length is W. Since the beam is wide (like a panel), E will be replaced by $E' = E/(1-\nu^2)$ in the usual elastica equations [12].

The pinned ends are pushed together until the beam buckles upward with span (base length) B. Then the ends are not allowed to deflect. At arc length S and time T, the horizontal coordinate is $X(S,T)$, the vertical coordinate is $Y(S,T)$, and the rotation is $\theta(S,T)$, as shown in Fig. 1. On the positive face of the cross section, the internal horizontal force is $P(S,T)$, positive if compressive, the internal vertical force is $Q(S,T)$, positive if downward, and the bending moment is $M(S,T)$, positive if counter-clockwise. Fig. 1 includes a concentrated weight W_0 at $S=C$.

Damping is neglected. The governing equations for the buckled beam are [13]

$$\frac{\partial X}{\partial S} = \cos\theta, \quad \frac{\partial Y}{\partial S} = \sin\theta,$$
$$E'I\frac{\partial\theta}{\partial S} = M, \quad \frac{\partial M}{\partial S} = Q\cos\theta - P\sin\theta,$$
$$\frac{\partial P}{\partial S} = -\left(\frac{W}{g}\right)\frac{\partial^2 X}{\partial T^2}, \quad \frac{\partial Q}{\partial S} = -W - \left(\frac{W}{g}\right)\frac{\partial^2 Y}{\partial T^2} \tag{1}$$

where g is the gravitational acceleration.

The numerical analysis is conducted in terms of nondimensional quantities, including

$$x = \frac{X}{L}, \quad y = \frac{Y}{L}, \quad s = \frac{S}{L}, \quad b = \frac{B}{L}, \quad c = \frac{C}{L}, \quad w = \frac{WL^3}{E'I},$$
$$w_0 = \frac{W_0 L^2}{E'I}, \quad r = \frac{W_0}{WL} = \frac{w_0}{w}, \quad p = \frac{PL^2}{E'I}, \quad q = \frac{QL^2}{E'I},$$
$$m = \frac{ML}{E'I}, \quad t = \left(\frac{T}{L^2}\right)\sqrt{\frac{E'Ig}{W}}, \quad \Omega = \omega L^2\sqrt{\frac{W}{E'Ig}} \tag{2}$$

where ω is a dimensional vibration frequency and r is the ratio of an attached weight to the total beam weight. The variables are written as the sum of equilibrium and dynamic components. The nonlinear equilibrium equations are solved first. Small free vibrations about equilibrium are considered, so that the dynamic equations are linearized in the dynamic components. Numerical results are obtained with the use of a shooting method involving subroutines NDSolve and FindRoot in Mathematica [13].

Due to symmetry of the buckled beam when there is no attachment, it is only necessary to consider attachments added symmetrically or on the left side (for instance) of the beam. For example, in Fig. 1, the nondimensional location of the attached weight from the left support is c, with $0<c<1$, and results for $0.5<c<1$ (right side) can be deduced from those for $0<c<0.5$ (left side).

3. Experiments

A thin polycarbonate strip was used for the experiments. The specific weight was 11.6 kN/m^3, the modulus of elasticity was 2.5 GPa, and Poisson's ratio was 0.37. The strip was 597.6 mm long between pinned ends, 152.4 mm wide, and 0.508 mm thick. Therefore $W = 0.898$ N/m and the nondimensional beam weight parameter was $w=44.2$. Three base lengths were used: 330.3 mm, 381.0 mm, and 431.8 mm (i.e., $b = 0.553, 0.638$, and 0.723). Attached weights were combinations of 7.12 g and 11.56 g (i.e., $r=0.130$ and 0.211).