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Robust identification of backbone curves using control-based continuation

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ABSTRACT

Control-based continuation is a recently developed approach for testing nonlinear dynamic systems in a controlled manner and exploring their dynamic features as system parameters are varied. In this paper, control-based continuation is adapted to follow the locus where system response and excitation are in quadrature, extracting the backbone curve of the underlying conservative system. The method is applied to a single-degree-of-freedom oscillator under base excitation, and the results are compared with the standard resonant-decay method.

1. Introduction

The constant drive to improve the performance of engineering structures is increasingly leading to lighter and more flexible designs where nonlinearity is inherent. While nonlinearity is often viewed as detrimental, recent contributions in the literature have shown it can actually be exploited for improving system performance. For instance, nonlinearity was deliberately introduced in the design of sharp acoustic switches and rectifiers [1], vibration absorbers [2] and energy harvesters [3]. The presence of nonlinearity however poses new challenges to engineers because, in contrast with linear systems, nonlinear systems can exhibit a wide variety of complicated dynamic phenomena such as intermittency, quasi-periodic oscillations, chaos, and bifurcations. A problem of particular interest to engineers is the prediction of the response of a system at resonance, where the system is at great risk of failure.

Pioneered in the 1960s by Rosenberg [4], the concept of nonlinear normal modes (NNMs) is considered as the natural extension of linear normal modes (LNMs) to nonlinear systems. Oscillations in nonlinear systems are energy-dependent such that the resonance frequency generally varies with the oscillation amplitude. NNMs can trace out this evolution, thereby generating the so-called backbone curve. For systems with moderate linear modal damping, the NNMs of the underlying conservative systems generally describe well the evolution of the resonance frequencies of the damped forced system. NNMs of conservative systems are defined as families of non-necessarily synchronous periodic oscillations [5]. There exist both analytical [6,7] and numerical [8,9] methods to calculate NNMs from a mathematical model. The latter are

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quite sophisticated, see for example the review [10], and can address complex real-life structures such as a full-scale spacecraft structure [11].

Experimental extraction of modal properties plays a central role in the updating and validation of linear structural models. In the context of nonlinear systems, the system's forced response and backbone curves can be used to estimate model parameters [12–14] and apply model updating techniques [15–17]. The experimental identification of NNMs was proposed in Refs. [18,19]. Following the principle of linear phase separation techniques, the method isolates a single NNM using an appropriate excitation. The applied force is then stopped and the relation between amplitude and frequency of oscillation is extracted from the free, damped, response of the system, a method often termed resonant decay. The method was successfully applied to several academic systems of moderate complexity as such, a single-degree-of-freedom oscillator [20], a nonlinear beam [19] and a steel frame structure [21]. A phase separation method where multiple NNMs are identified simultaneously from broad-band data was introduced and demonstrated on noisy synthetic data in Ref. [22].

The present paper proposes a novel method for extracting experimentally the underlying NNMs of a forced, damped system. The proposed method is robust to bifurcations and stability changes that arise in the tested system dynamics, and differs from existing contributions, such as resonant decay, in that the backbone curve is no longer post-processed from the measured data but rather directly traced out in the experiment. To this end, the steady-state periodic solutions that describe the backbone curve and a NNM of the underlying conservative system are followed for increasing vibration amplitudes using the control-based continuation (CBC) method. CBC combines stabilizing feedback control and path following techniques to explore the dynamics of a nonlinear system directly during the physical experiment, tracking the evolution of its steady-state response as system parameters are varied. The fundamental ideas underlying CBC were introduced by Sieber and Krauskopf in Ref. [23]. The first experimental demonstration of the method was performed on a parametrically excited pendulum whose periodic solutions were followed through a saddle-node bifurcation, beyond which point the solutions become unstable [24]. The frequency response of a harmonically excited impact oscillator was studied in Ref. [25] and Barton et al. investigated the periodic solutions of two energy harvesters in Refs. [26,27].

The paper is organized as follows: Section 2 reviews the connection that exists between NNMs and the forced response of a damped system. Section 3 discusses the identification of NNMs within an experiment. The so-called resonant-decay method currently used for extracting NNMs is first presented in Section 3.1. The CBC approach developed in this paper is then introduced and adapted to track the steady-state periodic solutions that define the backbone curve. A single-degree-of-freedom (SDOF) set-up is considered for demonstrating the proposed approach. It is presented in Section 4.1, and experimental results are discussed in Sections 4.2 and 4.3. A mathematical model of the SDOF oscillator is derived in Section 5 in order to further elaborate on the comparison between the proposed CBC approach and resonant decay. Conclusions are drawn in Section 6.

2. NNM motions in the presence of damping

Peeters et al. showed that a forced damped system can follow precisely one NNM motion of the underlying conservative (unforced) system provided that an appropriate excitation is applied [18]. Consider a multi-degree-of-freedom (DOF) system with stiffness nonlinearities, the equations of motion can be written as

$$\mathbf{M}\ddot{\mathbf{x}}+\mathbf{C}\dot{\mathbf{x}}+\mathbf{K}\mathbf{x}+\mathbf{f}_{\mathrm{nl}}(\mathbf{x})=\mathbf{f}(t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the linear mass, damping and stiffness matrices, respectively. The displacement, velocity and acceleration vectors are $\mathbf{x}$, $\dot{\mathbf{x}}$, and $\ddot{\mathbf{x}}$. The external loads are $\mathbf{f}(t)$. The nonlinear force vector is $\mathbf{f}_{\mathrm{nl}}(\mathbf{x})$ and it is assumed to contain odd nonlinearities only.

Consider the multi-point multi-harmonic excitation

$$\mathbf{f}(t)=\sum_{k=1}^{\infty}\mathbf{F}_k\,\sin(k\omega t), \tag{2}$$

where $\mathbf{F}_k$ corresponds to the fully populated amplitude vector of the force for the kth harmonic frequency. The system can be studied when the response is in quadrature with the excitation, i.e. when $\mathbf{x}(t)$ is 90° out of phase with $\mathbf{f}(t)$. With this assumption, $\mathbf{x}(t)$ can be decomposed as a cosine series

$$\mathbf{x}(t)=\sum_{k=1}^{\infty}\mathbf{X}_k\,\cos(k\omega t) \tag{3}$$

where $\mathbf{X}_k$ are real vectors. The nonlinear force term can be decomposed in a similar cosine series with coefficients $\mathbf{F}_{\mathrm{nl},k}$. Substituting Eqs. (2) and (3) into Eq. (1), each sine and cosine term in each harmonic can be balanced to give the following relations:

$$-k^2\omega^2\mathbf{M}\mathbf{X}_k+\mathbf{K}\mathbf{X}_k+\mathbf{F}_{\mathrm{nl},k}=0,\quad \forall k \tag{4}$$

$$-k\omega\mathbf{C}\mathbf{X}_k=\mathbf{F}_k, \tag{5}$$

These equations reveal that if the harmonics of the excitation are all in quadrature with the harmonics of the response, the