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On propagation of plane symmetric waves in a periodically corrugated straight elastic layer

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ABSTRACT

The classical Bernoulli – Euler model is employed for analysis of the plane wave propagation in an infinitely long elastic layer with a periodically varying thickness. The Floquet theory is applied to derive asymptotic formulas defining location and broadness of frequency stop bands for several corrugation shapes with different levels of discontinuity. For a layer with the perfectly smooth periodic corrugation, the equation of the axial wave motion is solved by the method of multiple scales in vicinity of critical frequencies. Then it is transformed to the canonical Mathieu equation, and its stability diagram is compared with predictions of the Floquet theory. The qualitative differences between the shapes of stop bands are discussed. The Wentzel – Kramers – Brillouin approximation is employed to solve the Bernoulli – Euler equation for the layer with perfectly smooth periodic corrugation, and the results are discussed in view of existence of pass and stop bands and in view of the levels of approximation used in alternative asymptotic methods.

1. Introduction

Analysis of wave propagation in infinite periodically corrugated waveguides is the classical subject carefully studied in many research papers. The existence of frequency stop bands is a remarkable feature of stationary dynamics of these structures, which may easily be detected by means of the Floquet theory [1]. The generation of stop bands is usually attributed to the destructive interference of incident and reflected waves at inhomogeneities, such as abrupt changes in geometry and/or material properties of components of a periodic waveguide. However, it has also been found out that stop bands emerge when the corrugation shape is perfectly smooth.

Although a broad range of mathematical models of periodic structures has been explored in the literature, and interesting periodicity-induced effects have been detected, it appears that a relatively simple issue of the 'sensitivity' of the location and the broadness of stop bands to the corrugation shape has not yet been fully addressed. The purpose of this paper is to fill in this gap by the analysis of a classical problem of propagation of the plane symmetric wave in a straight elastic layer with periodically varying thickness introduced in four alternative ways. The classical Bernoulli – Euler model is, probably, the ultimate simplification of this problem, which may be reliably used within the well-defined range of validity. Its simplicity facilitates the stop band analysis, and makes it possible to obtain the exact closed form solutions for certain corrugation shapes. Thus, it is meaningful to employ this model also for investigation of other issues, such as a link between the classical effect of temporal periodicity (e.g., the instability of a single-degree-of-freedom mechanical system under

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parametric excitation) and the stop band effect of the spatial periodicity. It enables application of the classical Ince – Strutt stability diagram for predictions of the location and the broadness of stop bands. Furthermore, the WKB (Wentzel – Kramers – Brillouin) approximation may also be readily used in the framework of the Bernoulli – Euler model of propagation of plane symmetric waves in a non-uniform layer. Then results obtained for an infinite periodic waveguide by means of the WKB approximation may be compared with predictions of the Floquet theory.

As is well-known, the propagation of low-frequency symmetric waves in a straight rod is adequately modelled by means of the elementary Bernoulli – Euler theory. Its asymptotic consistency with exact solutions for a planar layer of the constant thickness (Rayleigh – Lamb problem, see [2]) and for a cylindrical rod (Pochhammer – Chree problem, see [3]) suggests that this theory may be used for a layer with a moderately non-uniform thickness and for rods with moderately non-uniform cross-sections. Exact solutions for several types of variation of the cross-sectional area of a rod along its length are given in references [4,5].

Propagation of plane elastic waves in one-dimensional waveguides, often referred to as phononic crystals with functionally graded materials, has been extensively studied by many authors, and the references [6–9] are just a few examples. The analysis is usually conducted numerically by means of the transfer matrix method or the spectral element method. Application of these methods implies division of a periodic structure into an array of elements each of which has constant stiffness and inertia parameters. Therefore, the conventional wave equation with constant coefficients has been used in [6–9] to derive a transfer matrix or a spectral element. The present paper is concerned with the analytical solutions (exact or asymptotic) for alternative periodic corrugation shapes and, therefore, differs from the abovementioned references.

The problem of wave propagation in a two-dimensional waveguide with corrugated boundaries has been considered by many authors in Cartesian and cylindrical coordinates. Propagation of acoustic waves in a periodically corrugated planar duct has been studied by Nayfeh in [10] by means of the multiple scales method and, as shown in this paper, location of stop bands is determined by the resonances. Further development by Bostrom [11] has been concerned with the solution of a similar problem in cylindrical geometry with the special reference to an identification of the boundaries between stop and pass bands in the 'frequency–corrugation amplitude' plane. The similar work in elastodynamics (the Rayleigh – Lamb problem for an isotropic layer with corrugated thickness) has been done in [12,13]. In particular, El-Bahravy [12] has approximately found the boundaries between pass and stop bands for an elastic layer in the same way as it has been done in the Ref. [11].

More recently, the two-dimensional acoustical problem has been revisited in [14–16], and the two-dimensional problem in elasto-dynamics in [17–19]. Each of these papers has been concerned with the multi-modal response of a periodic waveguide. These references attribute the emergence of stop bands at the critical frequencies to the Bragg resonance phenomenon. In [14,17], the sinusoidal corrugation is considered, whereas in [18,19] the corrugation shape is rectangular (i.e., piece-wise constant). In [15], the sinusoidal corrugation of boundaries has different amplitudes, but the same spatial frequency and phase. In [19], two piece-wise constant boundary shapes are considered, both having the same spatial frequency, but with different phases. It is suggested in [15] that the gradients of boundary roughness might influence the properties of periodically corrugated waveguides. However, the solution reported in this reference is obtained for sinusoidal (i.e., continuous) shape of corrugation. To summarize the references [10–19], it appears that no direct comparison of shapes and locations of borders between stop- and pass bands has been conducted. The underlying reason for this state-of-the-art could be significant difficulties in finding a solution for such a generalized acoustical problem and Rayleigh – Lamb problem in elasto-dynamics.

The classical exposition of the Bragg scattering in a periodic medium can be found in many references with Lecture Notes [20] being just an example. Notably, in this reference the sinusoidal corrugation shape is employed for the analysis of one-dimensional scattering of elastic waves in a rod by means of the multiple scales method. This is exactly the model used in the present paper. As will be shown, it sets up a perfect stage to explore the role of smoothness of the corrugation shape in shaping up the boundaries between stop and pass bands, simply because this model facilitates derivation of exact solutions of governing equation for piece-wise constant and piece-wise linear corrugations. Although the standard tool for such an analysis is the Floquet theory, the multiple scales method will also be used hereafter, where appropriate.

The present paper goes along the same lines as the references [10–13] but differs from those in the mathematical model of a corrugated elastic layer, in a variety of alternative corrugation shapes and in a variety of the solution methods employed. It is structured as follows. In Section 2, the conventional Floquet analysis of wave propagation in an infinite periodic waveguide is done for three corrugation shapes and asymptotic formulas for the widths of stop band are derived. Section 3 is concerned with the asymptotic analysis of wave propagation in a layer with the perfectly smooth corrugation shape. The Bernoulli – Euler equation for propagation of plane symmetric waves is transformed to the canonical Mathieu equation. The stability diagram is compared with the location of pass and stop bands. In Section 4, the WKB approximation is employed to derive and solve the transport equation for the layer with perfectly smooth periodic corrugation of thickness. The unresolved questions are formulated and briefly discussed in Section 5. The results are summarized in view of their validity ranges in Conclusions.

2. Floquet analysis for three types of discontinuous corrugations

In this paper, the classical Bernoulli – Euler model of propagation of a free symmetric wave in an elastic layer under plane strain conditions is used. The governing equation for propagation of a plane longitudinal time-harmonic wave at the frequency ω in this non-uniform waveguide of the unit width and the variable thickness $h(x)$ is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[h(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right]+\frac{\omega^2}{c^2}h(x)u(x)=0 \tag{1}$$