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Natural frequencies of structures with interval parameters

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A B S T R A C T

This paper deals with the evaluation of the lower and upper bounds of the natural frequencies of structures with uncertain-but-bounded parameters. The solution of the generalized interval eigenvalue problem is pursued by taking into account the actual variability and dependencies of uncertain structural parameters affecting the mass and stiffness matrices. To this aim, interval uncertainties are handled by applying the *improved interval analysis via extra unitary interval* (*EUI*), recently introduced by the first two authors. By associating an *EUI* to each uncertain-but-bounded parameter, the cases of mass and stiffness matrices affected by fully disjoint, completely or partially coincident uncertainties are considered. Then, based on sensitivity analysis, it is shown that the bounds of the interval eigenvalues can be evaluated as solution of two appropriate deterministic eigenvalue problems without requiring any combinatorial procedure. If the eigenvalues are monotonic functions of the uncertain parameters, then the exact bounds are obtained. The accuracy of the proposed method is demonstrated by numerical results concerning truss and beam structures with material and/or geometrical uncertainties.

1. Introduction

The evaluation of the natural frequencies and the corresponding mode shapes plays a crucial role in vibration analysis since it provides a great deal of information concerning the dynamic characteristics of a system. Within a deterministic setting, this task is accomplished by solving the generalized eigenvalue problem which involves the mass and stiffness matrices of the structure. Changes of inertial and stiffness properties due to uncertainties inherent in any design process may affect to a large extent the vibration characteristics of a structural system. It is, therefore, of primary interest for design purposes to estimate the effects of geometrical and/or material uncertainties on the natural frequencies. Such uncertainties are commonly described within a probabilistic framework by using the random variable or random field concept. However, in the last decades, the so-called non-probabilistic approaches, such as convex model, fuzzy sets or interval model [1], have increasingly spread as alternative tools for handling uncertainties arising in engineering problems. The interval model, stemming from the *interval analysis* [2,3], is widely used when only the range of variability of non-deterministic properties is known but available data are insufficient to make reliable assumptions on the joint probability density function.

If the uncertain parameters are modeled as interval variables, the mass and stiffness matrices of the structure turn out to be interval matrices and the eigenvalue analysis leads to the so-called generalized or standard interval eigenvalue problems.

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The solution of these problems is a very difficult task since it consists of the evaluation of all possible eigenvalues and eigenvectors as the interval stiffness and mass matrices vary between their bounds. In practice, the objective is the determination of the narrowest intervals enclosing all possible eigenproperties, say the evaluation of the bounds of the eigenvalue and associated eigenvector for each eigensolution.

The solution of the interval eigenvalue problem has attracted much research attention in the last decades. Rohn [4] studied the generalized interval eigenvalue problem and derived formulas for the interval eigenvalues of a symmetric interval matrix with an error matrix of rank one. Based on the invariance properties of the characteristic vector entries, Deif [5] developed a method for the solution of the standard interval eigenvalue problem. The application of this method is limited by the lack of an efficient criterion for judging the invariance of signs of the eigenvectors components under interval matrix operations before computing interval eigenvalues. Under the assumption that the deviation amplitudes of the mass and stiffness matrices are positive semi-definite, Qiu et al. [6] proposed a procedure for the solution of the generalized interval eigenvalue problem which leads to two deterministic eigenvalue problems involving the bounds of the mass and stiffness matrices. The effectiveness of this method has been assessed by comparison with Deif's solution in the simplest case of fully disjoint mass and stiffness uncertainties. Following a similar reasoning, Elishakoff [7] proposed a procedure for finding the range of eigenvalues due to uncertain elastic moduli and mass density by using the upper and lower stiffness and mass matrices. A perturbation method for the solution of the generalized interval eigenproblem has been developed by Qiu et al. [8] by viewing the deviation amplitudes of the mass and stiffness matrices as perturbations around the nominal values of the interval matrix pair. The procedure is applicable for small deviation amplitudes and has been validated only in the case of fully disjoint mass and stiffness uncertainties. Qiu et al. [9] introduced the Eigenvalue Inclusion Principle (EIP) which leads to the solution of two deterministic eigenvalue problems as well. If the mass and stiffness matrices are affected by different uncertainties, the exact bounds are obtained. In general, this approach is accurate and efficient but it does not provide a physically consistent treatment of uncertainties affecting simultaneously the stiffness and mass matrices. Furthermore, the EIP is applicable only when the matrix pairs can be expressed by the non-negative decomposition. Based on a previously developed interval finite element method, Modares et al. [10] proved that, in the presence of any physically allowable uncertainty in the structural stiffness, the solutions of two deterministic eigenvalue problems are sufficient to obtain the exact bounds of the system's fundamental frequencies without resorting to any combinatorial solution procedure. Gao [11] proposed the interval factor method to investigate the effects of geometrical and material interval uncertainties on the natural frequencies and mode shapes of truss structures. Despite its simplicity, the method provides physically inconsistent results such as the independency of natural frequencies and mode shapes on the uncertainty of cross-sectional areas and Young's moduli, respectively. Furthermore, the dispersion of the interval eigenproperties around their midpoint values turns out to be unexpectedly independent of the mode order. Several perturbation-based (see e.g. [12–15]) or iterative procedures (see e.g. [16–19]) for the evaluation of the interval eigenvalue bounds have been also developed in the last decades. An evolution strategy for computing eigenvalue bounds of interval matrices has been presented by Yuan et al. [20]. In an attempt to take into account the dependencies of the uncertain parameters entering the mass and stiffness matrices, recently an approach based on a modified affine arithmetic has been proposed [21]. Besides the involved solution procedure, a common drawback of the aforementioned approaches is that their accuracy is assessed only for simple examples with fully disjoint mass and stiffness uncertainties.

The aim of this paper is to propose an efficient method for the solution of the generalized interval eigenvalue problem, able to overcome the limitations of available procedures discussed above. The key idea is to seek the bounds of the eigenvalues taking into account the actual influence of uncertainties on the mass and stiffness matrices and their dependencies. In other words, rather than tackling the problem from a merely mathematical point of view, the proposed procedure seeks a solution consistent with the physical behaviour of the structure. Interval uncertainties are handled following the *improved interval analysis via extra unitary interval* [22,23]. All possible situations occurring in real engineering problems, where uncertainties affecting the mass and stiffness matrices may be fully disjoint, completely or partially coincident, are examined. In each of these cases, a preliminary sensitivity analysis is performed in order to investigate the behaviour of the eigenvalues as functions of the uncertain parameters [1,24]. Based on the information provided by the eigenvalue sensitivities, the combinations of the extreme values of the uncertain parameters corresponding to the bounds of the eigenvalues are determined. Hence, the eigenvalue bounds can be evaluated as solution of two appropriate deterministic eigenvalue problems without any combinatorial procedure. This ensures substantial computational advantages over the *vertex method* [25] which yields the exact bounds of monotonic eigenvalues at the expense of the onerous solution of as many deterministic eigenvalue problems as are the combinations of the extreme values of the uncertain structural parameters.

The accuracy of the proposed procedure is demonstrated by analysing two truss structures and a FE modeled cantilever beam in the three cases of mass and stiffness matrices affected by fully disjoint, completely coincident and partially coincident uncertainties. It is demonstrated that the proposed estimates of the eigenvalue bounds are exact as long as the eigenvalues are monotonic functions of the uncertain parameters.

2. Problem formulation

2.1. Interval uncertainty modeling via extra unitary interval

The present study focuses on eigenvalue analysis of linear undamped structural systems with uncertain parameters, such as material and geometrical properties, affecting the mass and stiffness matrices. Within a non-probabilistic framework, uncertainties are represented as closed real interval numbers according to the so-called interval model. This model, mainly