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Interval-valued fuzzy threshold graph

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A B S T R A C T

Interval-valued fuzzy threshold graph is an extension of fuzzy threshold graph. In this paper, interval-valued fuzzy threshold graph, interval-valued fuzzy alternating 4-cycle, and interval-valued fuzzy threshold dimension are defined, and certain properties are studied. It is demonstrated that every interval-valued fuzzy threshold graph can be treated as an interval-valued fuzzy split graph.

1. Introduction

Threshold graph was first introduced by Chvatal and Hammer [4]. These graphs are used in several applied areas, such as psychology, neuro-science, computer science, artificial intelligence, and scheduling theory. The graphs can also be used to control the flow of information between processors, similar to how traffic lights are used in controlling the flow of traffic. Chvatal and Hammer introduced this graph to use in set-packing problems. In 1985, Ordman [13] used this graph in resource allocation problems. A graph $G=(V,E)$ is a threshold graph if there exists non-negative reals w_v $(v\in V)$ and t such that $W(U)\le t$ if and only if $U\subseteq V$ is an independent set, where $W(U)=\sum\limits_{v\in U} w_v$, t is called the threshold. So $G=(V,E)$ is a threshold graph whenever one can assign vertex weights such that a set of vertices is stable if and only if the total of the weights does not exceed a certain threshold. The threshold dimension $t(G)$ of a graph G is the minimum number k of threshold subgraphs $T_1,T_2,\ldots,T_k$ of G that cover the edge set of G. Threshold partition number, denoted by $tp(G)$, is the minimum number of edge disjoint threshold subgraphs needed to cover $E(G)$. An edge cover of a graph G is a set of edges $C\subset E$ such that each vertex is incident with at least one edge in C. The set C is said to cover the vertices of G.

Suppose that there are ATMs that are linked with bank branches. Each ATM has a threshold limit amount. This limit sets the maximum withdrawals per day. The goal is to determine a replenishment schedule for allocating cash inventory at bank branches to service a preassigned subset of ATMs. The problem can be modelled as a threshold graph since each ATM has a threshold of transactions. In reality, each ATM can have a different withdrawal limit. These withdrawal limits can be represented by an interval-valued fuzzy set. The threshold limit can be set such that the branches can replenish the ATMs without ever hampering the flow of transactions in each ATM. Motivated by this example, we investigate use of the interval-valued fuzzy threshold graph to model and solve this type of real problem.

Another graph related to the threshold graph is Ferrers diagraph. Ferrers diagraph was introduced by Peled and Mahadev [14]. A diagraph $\vec{G}=(V,\vec{E})$ is said to be a Ferrers diagraph if it does not contain vertices x,y,z,w, not necessarily distinct, satisfying $\overrightarrow{(x,y)},\overrightarrow{(z,w)}\in\vec{E}$ and $\overrightarrow{(x,w)},\overrightarrow{(z,y)}\notin\vec{E}$. For a diagraph $\vec{G}=(V,\vec{E})$, the underlying loop less graph $U(\vec{G})=(V,E)$, where $E=\{(u,v):u,v\in V,\ u\neq v,\ \overrightarrow{(u,v)}\in\vec{E}\}$. Graph theory has had great impact in the real world. After introduction of Euler's graph theory, Rosenfeld [26] generalizes the graph (crisp) as a fuzzy graph. Rosenfeld first considered the fuzzy relation between fuzzy sets and developed various theoretical graph concepts. The field of fuzzy graph theory is growing rapidly because of demands in nature. It has been used to solve such problems as human cardiac functions, fuzzy neural networks, routing problems, traffic light

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problems, and time table scheduling. In fuzzy mathematics, there are different types of fuzzy graphs. There can be graphs with crisp vertex set and fuzzy edge set or fuzzy vertex set and crisp edge set or fuzzy vertex set and fuzzy edge set or crisp vertices and edges with fuzzy connectivity, etc. Problems involving approximate reasoning use fuzzy systems. A *fuzzy set* α on a set X is denoted by $\alpha = (X, \mu)$ and is defined by a function $\mu : X \to [0, 1]$, called the membership function. There are many extensions of fuzzy sets, such as bipolar fuzzy set, interval-valued fuzzy set, and intuitionistic fuzzy set.

In 1971, Zadeh [40] introduced the notion of interval-valued fuzzy sets as a generalization of fuzzy sets in which the membership values are intervals of numbers instead of numbers. As the interval-valued fuzzy set is an interval number, it is more effective in solving uncertainty cases than traditional fuzzy sets. Therefore, it has broader applications, such as fuzzy control, approximate reasoning, medical diagnosis, intelligent control, and multivariate logic.

2. Review of previous research

In 1973, Chvatal and Hammer introduced the threshold graph. Bipartite threshold graphs were studied in [5] under the name difference graphs because they can equivalently be characterized as the graphs (V, E) for which there exist weights $w_v, v \in V$ and a real number t such that $|w_v| < t$ for every v and $(u, v) \in E \Leftrightarrow |w_u - w_v| > t$.

Andelic and Simic [2] studied properties of threshold graphs. Bhutani et al. [3] studied degrees of end nodes and cut nodes in fuzzy graphs. Makwana et al. [6] discussed the extraction of illumination invariant features using a fuzzy threshold based approach. Mathew and Sunitha [7] defined different types of arcs in a fuzzy graph. Mordeson and Nair [8] gave details of fuzzy graphs and hypergraphs. Pramanik [19] et al. found techniques to find the shortest path in an interval-valued fuzzy hypergraph. Recently, Pramanik et al. [16] defined and studied fuzzy ϕ-tolerance competition graphs. The fuzzy ϕ-tolerance competition graph is an extension of the fuzzy tolerance graph [28]. Samanta and Pal [31,32] worked on fuzzy planar graphs. The interval-valued fuzzy planar graph is a generalization of the fuzzy planar graph introduced in [15] by assigning each vertex and edge to interval-valued fuzzy numbers instead of traditional fuzzy numbers. The bipolar fuzzy hypergraph is a hypergraph in which each vertex and edge is assigned bipolar fuzzy sets. Samanta and Pal [36] have introduced bipolar fuzzy hypergraphs that are important in complex networking systems. Colouring is also a challenging problem. Samanta et al. [38] have found a colouring technique in the approximate sense of fuzzy graphs. The reader may find work on various extensions of fuzzy graphs in [1,21–25]. For further study of fuzzy graphs and variations the literature [17,18,20,29,30,33–35,37] may be very helpful. Nagoorgani and Radha [9] proved some results of regular fuzzy graphs. Nair and Cheng [10] represent cliques and fuzzy cliques in fuzzy graphs. Nair [11] defined perfect and precisely perfect fuzzy graphs. Natarajan and Ayyasawamy [12] described strong and weak domination in fuzzy graphs. Tao [39] et al. described image thresholding using graph cuts.

Multi-processor scheduling, bin packing, and the knapsack problem are variations of set-packing problems; a well-studied problem in combinatorial optimization. These problems have had a large impact on the design and analysis of interval-valued fuzzy threshold graphs. All of these problems involve packing items of different sizes into bins with finite capacities. Consider a parallel system consisting of a set of independent processing units each of which has a set of time-sharable resources, such as a CPU, one or more disks, and network controllers. Here all units have variable capacities as well as resources. Motivated by the problem of packing independent units by optimizing capacities, interval-valued fuzzy threshold graph is introduced. This parallel system can be described by interval-valued fuzzy threshold graph where each of the units and resources represents the vertices of the graph and a task executing on one of the units places requirements on each of the resources that can be best described by edges.

In this paper, we study several properties of an interval-valued fuzzy threshold graph as an extension of a fuzzy threshold graph. We also define, for instance, interval-valued fuzzy split graphs and interval-valued fuzzy threshold dimensions.

3. Preliminaries

A *graph* $G = (V, E)$ consists of a set denoted by V, or by $V(G)$ and a collection E, or $E(G)$, of unordered pairs (u, v) of elements from V. Each element of V is called a *vertex* or a point or a node, and each element of E is called an *edge* or an arc or a line or a link.

An *independent set (stable set)* in a graph $G = (V, E)$ is the set of vertices of V, no two of which are adjacent. That is, $S(\subset V)$ is said to be an independent set if for all $u, v \in S$ $(u, v) \notin V$. The *maximal independent set* is an independent set in which adding any other vertex, causes the new set to fail to be independent. The *maximum independent set* is the maximal independent set with the largest number of vertices.

A *fuzzy set* α on a set X is a mapping $\alpha : X \to [0, 1]$, called the *membership function*. The *support* of α is $\mathrm{supp}(\alpha) = \{x \in X | \alpha(x) \neq 0\}$ and the *core* of α is $\mathrm{core}(\alpha) = \{x \in X | \alpha(x) = 1\}$. The *support length* is $s(\alpha) = |\mathrm{supp}(\alpha)|$ and the *core length* is $c(\alpha) = |\mathrm{core}(\alpha)|$. The height of α is $h(\alpha) = \max\{\alpha(x) | x \in X\}$. The fuzzy set α is said to be *normal* if $h(\alpha) = 1$.

A *fuzzy graph* with a non-empty finite set V as the underlying set is a pair $G = (V, \sigma, \mu)$, where $\sigma : V \to [0, 1]$ is a fuzzy subset of V and $\sigma : V \times V \to [0, 1]$ is a symmetric fuzzy relation on the fuzzy subset σ such that $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\wedge$ stands for minimum. A fuzzy edge (x, y), $x, y \in V$ is said to be strong [35] if $\mu(x, y) \geq \frac{1}{2}\min\{\sigma(x), \sigma(y)\}$ and is called weak, otherwise.

A *fuzzy path* ρ in a fuzzy graph is a sequence of distinct nodes $x_0, x_1, x_2, \cdots, x_n$ such that $\mu(x_{i-1}, x_i) > 0$, $1 \leq i \leq n$. Here $n \geq 0$ is called the *length of the path*. The consecutive pairs (x_{i-1}, x_i) are called the *fuzzy arcs of the path*. The path ρ is a *fuzzy cycle* if $x_0 = x_n$ and $n \geq 3$. The fuzzy graph without a cycle is called *acyclic fuzzy graph* or *fuzzy forest*.

The *strength of connectedness* between two vertices u and v is $\mu^{\infty}(u, v) = \sup\{\mu^k(u, v) \big| k = 1, 2, \cdots\}$ where $\mu^k(u, v) = \sup\{\mu(u, u_1) \wedge \mu(u_1, u_2) \wedge \ldots \wedge \mu(u_{k-1}, v) | \; u_1, u_2, \ldots, u_{k-1} \in V\}$. In a fuzzy graph an arc (u, v) is said to be a *strong arc* [7] or *strong edge*, if $\mu(u, v) \geq \mu^{\infty}(u, v)$ otherwise it is *weak*.

Two vertices (nodes) in a fuzzy graph are said to be *fuzzy independent* if there is no strong arc between them. The set of all vertices that are mutually (fuzzy) independent is called the *fuzzy independent set*.

An *interval number* D is an interval $[a^-, a^+]$ with $0 \leq a^- \leq a^+ \leq 1$. For two interval numbers $D_1 = [a_1^-, a_1^+]$ and $D_2 = [a_2^-, a_2^+]$ the following are defined:

$$\text{i) } D_1 + D_2 = \left[a_1^-, a_1^+\right] + \left[a_2^-, a_2^+\right] = \left[a_1^- + a_2^- - a_1^- \cdot a_2^-, a_1^+ + a_2^+ - a_1^+ \cdot a_2^+\right],$$

$$\text{ii) } \min\{D_1, D_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right],$$