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Can a discrete dynamic model ever perfectly simulate a continuum?

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ABSTRACT

This article elaborates on the seemingly impossible notion that a continuous elastic body subjected to dynamic sources on its outer surface could ever be substituted in all of its essential qualities by a discrete model accomplished with finite elements. This topic is taken up herein and discussed in the context of a very simple two-dimensional model involving the propagation of SH shear waves (or acoustic waves) in a homogeneous elastic half-space. It is shown that there exists at least one discrete solid, referred to here as the *Guddati Solid*, which from its external surface behaves exactly like the continuum and is able to transmit waves of any frequency and any wavelength. This is a rather surprising finding in that it seems to contradict some well-known elastodynamic representation theorems, not to mention falsify the widespread belief that a discrete system can *never* behave like the continuum it purports to model. The purpose of this article is thus to present one example which disproves this widely believed postulate.

1. Introduction

It seems clear that with the steadily increasing sophistication of discrete models for elastodynamic problems in engineering and in science, the accuracy and behavior of numerical models has steadily increased, to the point that a refined model today may produce results that are hardly distinguishable from the real thing, that is, from the continuum that it replaces. But whatever the precision, those models are still not 100% exact, that is, they maintain a small even if negligible error in the solution, so in principle they attain the exact solution only in the ideal limit of an infinitesimal grid.

There exist also some fundamental principles in linear elastodynamics commonly referred to as *representation theorems* which make statements about the allowable states of stress- and deformation fields within an elastic body when it is subjected to sources applied somewhere. These are in turn closely related to the Principle of Virtual Displacements, the reciprocity principles of Maxwell-Betti, the Beltrami and Rayleigh principles, and other theorems such as the Somigliana Integral equation that lie at the heart of the Green's functions formalism. For example, these theorems play an important role in the theory of the Boundary Element Method (BEM) and in the assessment of the seismic motions at some distance from a causative earthquake fault. In essence, these representation theorems state that when an elastic body surrounded by some well-defined boundary is subjected to dynamic excitations onto its outer skin in the form of prescribed tractions or prescribed displacements — or both — the displacement field everywhere is unique and depends on the shape of the body and its material configuration.

Indeed, if we knew the so-called Green's functions (displacements due to unit point loads) for any position and direction of an applied point source on that surface, then we could just as well predict the motions anywhere for any arbitrary combination of surface tractions and displacements, and we would accomplish that by means of the representation theorems alluded to, or at least we could do so in principle. Change the solid without changing the size and characteristics of the surface and the expressions for the Green's functions will change accordingly, and with that, the predicted displacements will change everywhere. In the light of these technical considerations, it would seem that if the continuous solid were to be replaced by a discrete model, it could not possibly behave quite in the same fashion as the continuum. In particular, it would seem unlikely that a discrete model —no matter how smart and detailed — could ever replace the original continuous solid *exactly* in all of its essential aspects. For one, it could be expected that the discrete model might not even begin to transmit the full frequency spectrum contained in an arbitrary excitation in time.

But strange as it may seem, it turns out that in a series of extraordinary yet admittedly abstruse papers penned by Guddati et al [1–4] on the topics of *One-Way Wave Equations* and *Transmitting Boundaries*, they come up with a remarkable discrete solid that exhibits this seemingly impossible characteristic, namely that from its external surface, the discrete model feels and looks like the real continuous thing it replaces, and it does so for any load combination on its surface, even if not in its interior. At first this seemed so alien and strange to us that we had to thoroughly check out the facts, and we thus determined that it

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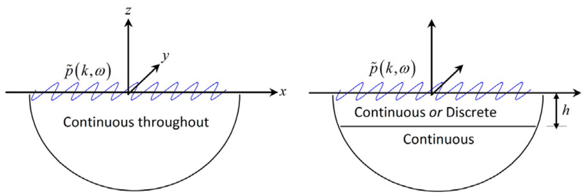

Fig. 1. : Homogeneous elastic half-space vs. semi-discrete thin layer representation. Left: Continuous, homogeneous half-space. Right: Half-space overlain by an arbitrarily thick layer with the same material properties as the half-space. The layer can be either continuous or discrete.

was indeed true. Thinking that a broad readership might also enjoy learning about this strange contraption, we decided to prepare this brief article providing a succinct and transparent outline on the particulars of this case, and doing so while avoiding needless complications.

2. Continuous layer underlain by continuous half-space

Consider first a continuous, homogeneous elastic half-space ($z \le 0$) subjected to anti-plane (SH) dynamic loads (or sources) $p_y = p(x, z = 0, t)$ applied at its upper free surface ($z = 0$). These loads can have any *arbitrary* spatial and/or temporal variation, including very high frequencies. The half-space responds in turn to these load with motions $u_y = u(x, z, t)$. The origin of coordinates is placed at the surface and the horizontal axis is defined in the full interval $[-\infty < x < \infty]$, see Fig. 1 on the left. The material properties of the half-space are the shear modulus μ, the mass density ρ, and the shear wave velocity $\beta = \sqrt{\mu/\rho}$. When this problem is formulated in the frequency-wavenumber domain, i.e. after a spatial-temporal Fourier transform is applied on both $p(x, 0, t)$and $u(x, 0, t)$, it is found that the surface tractions and observed displacements change into $\tilde{p}(k, \omega)$, $\tilde{u}(k, \omega)$ respectively, where k is the horizontal wavenumber and ω is the frequency. Furthermore, it can also readily be shown [5] that in *that* domain, the tractions and displacements on the surface of the non-discretized, continuous half-space (Fig. 1, left) are related by an *exact* stiffness or impedance function Z_H that depends on frequency and wavenumber. That is, we find that these quantities are related as:

$$\tilde{p}(k, \omega) = Z_H\, \tilde{u}(k, \omega), \quad Z_H = \mu\sqrt{k^2 - k_S^2}, \quad k_S = \omega/\beta \tag{1}$$

Formally, the *exact* solution at the surface follows then from the inverse Fourier transform

$$u(x, 0, t) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\tilde{p}(k, \omega)\mathrm{e}^{\mathrm{i}(\omega t - kx)}dk\, d\omega}{Z_H(k, \omega)} \tag{2}$$

Depending on the load characteristics (Fig. 1, left), this problem will generate shear waves that propagate laterally and downwards in all directions and leak energy into the abyss underneath. For example, a line load at $\mathbf{x} = (0, 0)$ will elicit a displacement field $\tilde{u}(x, z, \omega) = \frac{1}{2\mathrm{i}\mu}H_0^{(2)}(\omega\sqrt{x^2 + z^2}/\beta)$, [5]. The solution posed by Eq. (2) is *exact* in the sense that neither k nor ω are restricted in any way, and the half-space can transmit waves of any wavelength, any frequency and in any direction with respect to the vertical. That is, Eq. (2) provides the *exact* motion at the surface of the half-space for loads with *any* spatial or temporal variation. Observe also that Z_H does not contain any characteristic length.

We now proceed to overlay the half-space with a homogeneous, continuous layer of the *same* material properties and of *arbitrary* thickness h, as shown in Fig. 1 on the right. If we also denote with indices 1 and 2 the upper and lower bounding surfaces, then that layer, on its own, would be characterized by an *exact* dynamic stiffness matrix of the form [5]

$$\mathbf{Z}_{L\text{exact}} = \begin{Bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{Bmatrix} = ks\mu \begin{Bmatrix} \coth ksh & -\sinh^{-1} ksh \\ -\sinh^{-1} ksh & \coth ksh \end{Bmatrix},$$
$$ks = \sqrt{k^2 - k_S^2} \tag{3}$$

The complete system would ultimately be described by the equilibrium equation

$$\begin{bmatrix} \tilde{p}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} + Z_H \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} \tag{4}$$

Formally, we can now proceed to eliminate the auxiliary interface at the bottom of the surface layer by means of condensation, after which we obtain

$$\tilde{p}_1 = [Z_{11} - Z_{12}(Z_{22} + Z_H)^{-1}Z_{21}]\tilde{u}_1 = Z_H\tilde{u}_1 \tag{5}$$

As should be noticed, we have recovered the exact impedance of the half-space, because the *continuous* layer in this case has exactly the same material properties as the *continuous* half-space underneath. Thus, the systems on the left and right of Fig. 1 produce *exactly* the same result everywhere, as could have been expected.

3. The strange case of the Guddati solid

Although Guddati et al [1–4] — henceforth abbreviated collectively as G&a — considered in their papers a series of far more complex and abstruse problems than the one being summarized herein, for the purposes of this article it is enough to consider a simplified model that still abstracts all of the essential qualities of what we refer to as a *Guddati solid*. This solid is composed of one or more horizontal thin layers of the same material properties which have all been discretized in the finite element sense along the vertical direction —and in that direction only. Thus, G&a replaced the continuous layer of the previous section with a set of *discrete* layers, and coupled these to a *continuous* elastic half-space underneath. Fig. 1 on the right shows the situation with just one (at first thin) discrete layer of thickness h. This strategy of discretizing the layers is commonly referred to as the *Thin Layer Method* — or TLM for short — and it offers numerous advantages in the solution of wave propagation problems in vertically inhomogeneous media, such as plates or layered soils. Now, instead of using the classical formulation of the TLM, G&a used a weighted residuals formulation based on the midpoint integration rule in the context of a linear interpolation function. It is important to add that the standard TLM formulation that is *not* based on the midpoint rule fails to exhibit the remarkable property to be described. Although the material properties are again the same as those of the half-space, the discrete layer by itself is now characterized by a discrete impedance matrix that is given by [1] (compare with Eq. (3)):

$$\mathbf{Z}_L = \begin{bmatrix} a & b \\ b & a \end{bmatrix} = k^2\frac{\mu h}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{\mu}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2\frac{\rho h}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{6}$$

After Eq. (6) is used in lieu of Eq. (3) to form the global impedance matrix, we are led to the system equation in $\omega - k$ space for a single TLM layer underlain by an elastic half-space and subjected to a load $\tilde{p}_1(k, \omega)$ applied on the upper surface given by

$$\begin{bmatrix} \tilde{p}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ b & a + Z_H \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} \tag{7}$$

where a, b are trivially inferred from Eq. (6). The condensed stiffness is now (compare with Eq. (5))

$$Z_{eq} = a - b(a + Z_H)^{-1}b \tag{8}$$

Astonishingly, the above expression yields $Z_{eq} = Z_H$ for *any* frequency and *any* wavenumber, despite the fact that the thin layer is unable to propagate arbitrarily short waves and does also exhibit rather clear and well defined dispersion characteristics [6]. In addition, and perhaps even more remarkably, the condensed impedance is