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Reliability sensitivity estimation with sequential importance sampling

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ABSTRACT

In applications of reliability analysis, the sensitivity of the probability of failure to design parameters is often crucial for decision-making. A common sensitivity measure is the partial derivative of the probability of failure with respect to the design parameter. If the design parameter enters the definition of the reliability problem through the limit-state function, i.e. the function defining the failure event, then the partial derivative is given by a surface integral over the limit-state surface. Direct application of standard Monte Carlo methods for estimation of surface integrals is not possible. To circumvent this difficulty, an approximation of the surface integral in terms of a domain integral has been proposed by the authors. In this paper, we propose estimation of the domain integral through application of a method termed sequential importance sampling (SIS). The basic idea of SIS is to gradually translate samples from the distribution of the random variables to samples from an approximately optimal importance sampling density. The transition of the samples is defined through the construction of a sequence of intermediate distributions, which are sampled through application of a resample-move scheme. We demonstrate effectiveness of the proposed method in estimating reliability sensitivities to both distribution and limit-state parameters with numerical examples.

1. Introduction

In reliability analysis, the interest is in the assessment of the performance of an engineering system through evaluating its probability of failure. Let $\boldsymbol{X}$ denote a continuous random vector of dimension n modeling the system variables that are expected to present an uncertain behavior. The failure event can be defined as the collection of the outcomes of $\boldsymbol{X}$ for which the so-called limit-state function (LSF) $g(\boldsymbol{x})$ takes non-positive values.

Consider now a vector θ that collects deterministic parameters that enter the definition of the reliability problem. The vector θ may contain parameters of the joint probability density function (PDF) $f(.)$ of $\boldsymbol{X}$, denoted by θ^d, as well as parameters of the LSF $g(.)$, denoted by θ^g. Hence the determinstic parameter vector can be decomposed as $\theta = [\theta^d;\theta^g]$. The probability of failure can be expressed as follows:

$$P_f(\theta) = \int_{g(\boldsymbol{x},\theta^g)\leqslant 0} f(\boldsymbol{x}, \theta^d)d\boldsymbol{x} \tag{1}$$

Importantly, the LSF often depends on a computationally intensive numerical model of the engineering system so that evaluation of the integral in Eq. (1) becomes a nontrivial task. To alleviate computational requirements, a variety of tailored approaches have been developed [1,2]. These include approximation methods such as the first/second order reliability method (FORM/SORM) [3], response surface approaches [4] and simulation techniques based on the Monte Carlo method. All these methods have their merits and disadvantages as discussed in the literature (e.g. [5–8]).

In this contribution we focus on sampling-based methods. The Monte Carlo method is a simple and robust technique, that is able to handle any LSF, independent of its complexity. The efficiency of the standard Monte Carlo method does not depend on the dimension of the random variable space. The probability integral of Eq. (1) can be expressed as the expectation of the indicator function $I(g(\boldsymbol{x}, \theta^g) \leqslant 0)$, where $I(g(\boldsymbol{x}, \theta^g) \leqslant 0) = 1$ if $g(\boldsymbol{x}, \theta^g) \leqslant 0$ and $I(g(\boldsymbol{x}, \theta^g) \leqslant 0) = 0$ otherwise. Standard Monte Carlo estimates $P_f(\theta)$ for a given parameter vector θ by generating n_s independent samples $\{\boldsymbol{x}^{(k)}, k = 1, \ldots, n_s\}$ from the PDF $f(\boldsymbol{x}, \theta^d)$ and taking the sample mean of $I(g(\boldsymbol{x}^{(k)}, \theta^g) \leqslant 0)$, i.e.

$$\widehat{P}_f = \widehat{\mathrm{E}}_f[I(g(\boldsymbol{x}, \theta^g) \leqslant 0)] = \frac{1}{n_s}\sum_{k=1}^{n_s} I(g(\boldsymbol{x}^{(k)}, \theta^g) \leqslant 0) \tag{2}$$

The estimate of Eq. (2) is unbiased and has coefficient of variation:

$$\delta_{\widehat{P}_f} = \sqrt{\frac{1-P_f(\theta)}{n_s P_f(\theta)}} \tag{3}$$

$\delta_{\widehat{P}_f}$ is a measure of the statistical accuracy of $\widehat{P}_f$. According to Eq. (3), crude Monte Carlo requires approximately 10^{k+2} samples to achieve an accuracy of $\delta_{\widehat{P}_f} = 10\%$ for a probability in the order of 10^{-k}. Hence, the

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computational cost of crude Monte Carlo becomes intractable for low target failure probabilities. Several methods have been proposed that aim at enhancing the efficiency of the crude Monte Carlo method through decreasing the variance of the probability estimate. These include importance sampling (IS) and its adaptive variants [9–12], line sampling (LS) [13,14], subset simulation (SuS) [15,16] and sequential importance sampling (SIS) [17,18].

In many practical applications of reliability analysis, one is interested in understanding the influence of each component of the vectors $\boldsymbol{X}$ and θ on the probability of failure. The influence of the random variables $\boldsymbol{X}$ can be quantified by a variety of sensitivity analysis approaches, e.g. the FORM sensitivity indices [3] or other variance-based sensitivity measures (e.g. [19–21]). The sensitivity to deterministic parameters θ can be quantified through performing parameter studies for a range of parameter values or through evaluating the local partial derivative derivatives of $P_f(\theta)$ to the components of θ. In this paper, we focus on efficient estimation of the latter.

Approximations of the partial derivative of $P_f(\theta)$ in terms of both distribution parameters θ^d and limit-state parameters θ^g are obtained as a byproduct of FORM/SORM solutions [1,22]. They involve the evaluation of first- or second-order derivatives of the isoprobabilistic transformation to the independent standard normal space and the derivative of the limit-state function $g(.)$ to the respective parameters at the most probable failure point.

The derivative of the probability of failure $P_f(\theta)$ with respect to a distribution parameter θ_i^d is obtained through differentiating the integrand of Eq. (1) as follows:

$$\frac{\partial P_f(\theta)}{\partial \theta_i^d} = \int_{g(\boldsymbol{x},\theta^g)\leqslant 0} \frac{\partial f(\boldsymbol{x},\theta^d)}{\partial \theta_i^d} d\boldsymbol{x} \tag{4}$$

The expression in Eq. (4) is a (possibly high dimensional) domain integral and can be estimated using any reliability method including standard Monte Carlo methods. This has been acknowledged by Wu in [23], who estimated distribution parameter sensitivities with standard IS. Other variance reduction methods can also be applied, such as SuS [24,25] and adaptive IS based on surrogate models [26]. However, it should be noted that these approaches cannot be implemented in a transformed random variable space, which is commonly employed for estimating $P_f(\theta)$. This can be understood by considering that when transforming the random variable space to an equivalent standardized space (e.g. the independent standard normal space), distribution parameters become parameters of the transformed LSF, while the standardized distribution is parameter-free. As discussed next, reliability sensitivities to limit-state parameters are not expressed as domain integrals. We note that several sampling-based reliability methods benefit from such a transformation; e.g. it facilitates the choice of appropriate IS functions [2] and it can provide a basis for dimensional-independent performance of sequential sampling methods [16,27,18].

A different approach was introduced in [19] for the estimation of distribution parameter sensitivities. The authors constructed a linear response surface using Monte Carlo samples and applied the FORM sensitivity results to this surrogate model. It is noted that this approach can also be applied to evaluate sensitivities to limit-state parameters, since the latter are also provided as byproducts of FORM.

Consider the case where the LSF $g(\boldsymbol{x},\theta^g)$ is continuously differentiable and $\nabla_{\boldsymbol{x}} g(\boldsymbol{x},\theta^g) \neq \boldsymbol{0}$ for all $\boldsymbol{x}$ and θ^g on the surface $\{g(\boldsymbol{x},\theta^g)=0\}$. The derivative of $P_f(\theta)$ with respect to a parameter of the LSF θ_i^g is then given by the following surface integral [22]:

$$\frac{\partial P_f(\theta)}{\partial \theta_i^g} = -\int_{g(\boldsymbol{x},\theta^g)=0} f(\boldsymbol{x},\theta^d)\frac{1}{\|\nabla_{\boldsymbol{x}} g(\boldsymbol{x},\theta^g)\|}\frac{\partial g(\boldsymbol{x},\theta^g)}{\partial \theta_i^g} ds(\boldsymbol{x}) \tag{5}$$

where $ds(\boldsymbol{x})$ denotes surface integration over the surface $\{g(\boldsymbol{x},\theta^g)=0\}$. Eq. (5) is a surface integral and cannot be estimated with classical Monte Carlo methods. However, one can estimate Eq. (5) through application of simulation methods that monitor the boundary of the limit-state function. Such methods are the directional sampling [28,29] and LS [13,14] methods. Both approaches require the solution of a line search problem for each sample to determine the intersection of the sampling direction or line with the limit-state surface. Application of these methods to reliability sensitivity is discussed in [29–32].

Alternative methods for solving the problem of Eq. (5) have been proposed in [33] and in [34]. The procedure introduced in [33] requires solving the limit-state equation $g(\boldsymbol{x},\theta^g)=0$ for one component of $\boldsymbol{x}$. Estimation of the reliability sensitivity is achieved by conditional IS in terms of the remaining random variables. The method presented in [34] involves a linear approximation of the LSF in terms of the parameter vector and an approximation of the probability of failure in terms of the LSF. The approximations are constructed perturbing the limit-state parameters at the samples close to the failure surface and performing additional LSF evaluations.

In [31], the authors introduced an approximation of the surface integral of Eq. (5) with a domain integral. This approach allows application of any simulation method for estimating Eq. (5). The same approximation was independently proposed in [35]. Therein, the domain integral approximation is estimated with SuS. In [36] this approach was combined with a polynomial chaos surrogate for efficient approximate reliability sensitivity analysis. In this contribution, we propose estimation of the domain integral approximation with SIS [18]. The basic idea of SIS is to gradually translate samples from the distribution of the random variables to samples from an approximately optimal importance sampling density. The transition of the samples is defined through the construction of a sequence of intermediate distributions. We demonstrate that this sequence is particularly suitable for estimating the domain integral approximation of Eq. (5). The proposed approach can also be applied for estimating Eq. (4) if estimation is performed at the independent standard normal space.

The structure of the paper is as follows. In Section 2, we review the approximation of reliability sensitivity to limit-state parameters through a domain integral. Section 3 discusses estimation of the approximation with SIS. Section 4 focuses on application of the proposed approach to evaluation of reliability sensitivity of system problems with multiple failure modes. Section 5 tests the proposed method with a series of numerical examples. The paper closes with the conclusions in Section 6.

2. Approximate reliability sensitivity analysis

In this section, we derive an approximation of the surface integral that arises in reliability sensitivities to limit-state parameters through a domain integral. This approximation, originally introduced in [31], enables estimation of the surface integral with standard Monte Carlo methods. We focus on sensitivities to limit-state parameters, i.e. the case where $\theta = \theta^g$. We note that if the reliability problem is recast in a standardized probability space through performing an isoprobabilistic transformation (e.g. [37]), distribution parameters become limit-state parameters. Therefore, in such settings the methods described here can be applied to the computation of reliability sensitivities to any deterministic parameter.

2.1. Approximation of the surface integral

We derive the approximation of the surface integral of Eq (5) through considering the following definition of the probability of failure:

$$P_f(\theta) = \int_{\mathbb{R}^n} I(g(\boldsymbol{x},\theta)\leqslant 0) f(\boldsymbol{x}) d\boldsymbol{x} \tag{6}$$

where $I(g(\boldsymbol{x},\theta)\leqslant 0)$ is the indicator function of the failure domain and $g(\boldsymbol{x},\theta)$ is a LSF that is continuously differentiable almost everywhere with respect to any component θ_i of θ. In the above expression, the dependence of parameters θ enters in the integrand. Therefore, one