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Quantum-inspired Boolean states for bounding engineering network reliability assessment

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ABSTRACT

Significant methodological progress has taken place to quantify the reliability of networked systems over the past decades. Both numerical and analytical methods have enjoyed improvements via a host of advanced Monte Carlo simulation strategies, state space partition methods, statistical learning, and Boolean functions among others. The latter approach exploits logic to approximate network reliability assessments efficiently while offering theoretical error guarantees. In parallel, physicists have made progress modeling complex systems via tensor networks (TNs), particularly quantum many-body systems. Inspired by the representation power of quantum TNs, this paper offers a new approach to efficiently bound all-terminal network reliability (ATR). It does so by exactly solving a related network Boolean satisfiability counting problem (or $\#SAT_{NET}$), represented as a TN, by counting configurations in which all network nodes are connected to at least a neighbor. Our $\#SAT_{NET}$ counting outperforms state-of-the-art approximate and exact counters for the same problem as shown for challenging two-dimensional lattice networks of increasing size. While the over-counting from $\#SAT_{NET}$ increases exponentially relative to the number of configurations that satisfy ATR problems (or $\#REL_{AT}$), the bias is predictable for ideal networks, such as lattices, and our upper-bound is guaranteed with 100% confidence. This bound also cuts through the upper bound from other analytical methods for the ATR problem, such as recursive decomposition algorithms (RDA)—a desirable feature when exact or approximate methods with error guarantees fail to scale. Clearly, our goal is not to solve the general stochastic network reliability problem, which remains a $\#P$ problem in the computational complexity hierarchy (i.e., a counting version of non-deterministic polynomial time [NP] problems for which there is no known polynomial time algorithms to find their solutions). Instead, we present a novel bounding technique for network reliability, which relies on exactly counting configurations for our network satisfiability problem by using quantum computing principles. We offer a proof for the counting bound to hold in a connectivity ATR sense, and illustrate trends with cubic and lattice networks. Overall, the proposed method provides an alternative to available system reliability assessment approaches, and opens directions for future research, especially as discoveries in logic, algebraic projections, and quantum computation continue to accrue.

1. Introduction

With the advent of community resilience principles in engineering [1–4], it is critical to advance methods to quantify the performance of systems, especially when abstracted as networks as in the case of distributed infrastructure. Recent efforts have focused on the reliability side that contributes to resilience. These include smarter simulation and graphical methods, particularly those based on Markov-Chain Monte Carlo (MCMC) [5,6], Bayesian Networks (BN's) [7,8], and statistical learning [9]. Also, closed-form methods and their approximations continue to offer unparalleled insights to system reliability problems, although these often require customized treatment to make them tractable [10–14].

The goal of the present study is to show an alternative approach for system reliability assessment that is rigorous and will have practical appeal as quantum computation and the requisite science and engineering developments consolidate. The new perspective relies on creating quantum tensor networks (TNs) that mirror general stochastic networked systems, but whose contraction (i.e., tensor products over all shared indices) yields a scalar $\#SAT_{NET}$ with the true count of

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configurations that satisfy a related Boolean problem—a superset of all satisfiable all-terminal reliability (ATR) configurations that counts configurations in which all nodes have at least one neighbor. As engineering networks typically do not contain hyperedges (links that join more than two end nodes) and have low degree (the number of links per node tend to be independent of network size), it is possible to perform the TN contractions efficiently and exactly via low-rank tensor products.

Recent research has linked quantum tensor networks to general *#SAT* [15,16], where computations are shown to remain generally *#P* if pursued with classical computers. If quantum computers are used instead, TN contractions yielding *#SAT* values for general networked systems (including hyperedges and non-constant node degree) can be approximated with additive error in polynomial time [17]. Using classical computers and algorithms, *#SAT* and network reliability have been linked recently, offering in polynomial time multiplicative guaranteed approximations to the counting of reliable configurations *#REL* [18]. However, no research has established a connection between TNs, which encode quantum Boolean states, and network reliability directly, especially for exact bounding purposes. We establish such a bound using quantum TNs that over-count all-terminal reliable configurations $\#REL_{AT}$, by counting satisfiable configurations $\#SAT_{NET}$ of the Boolean network representation, particularly for networks that capture engineering topologies and links are stochastic. This proposed approach provides new algorithmic alternatives to engineering reliability analysts and decision makers.

Each of the TN and satisfiability (*SAT*) foundational concepts are introduced below. We start with TNs, as they provide the algebraic method to compute properties of high-dimensional systems, including Boolean formulae, via tensor operations. Then, we present the *SAT* problem with example logical formulae. These two concepts combined, enabled us to upper-bound all-terminal network reliability efficiently. This is because defining a *SAT* formula for an engineered network is not difficult, and contracting such a formula via TN is efficient. The caveat is that the result from the TN contraction over-counts the reliable configurations of the ATR problem, thus allowing us to only provide an upper bound (which is nonetheless exact).

1.1. Algebraic tensor networks (TN)

At their simplest, tensor networks are linked multidimensional arrays, where the rank of the tensors corresponds to the number of indices in them, such as a rank-2 tensor, which corresponds to the familiar matrix $A_{\alpha\beta}$ in two dimensions α and β [19]. Linkages among tensors are important as they enable contractions over shared indices. For instance, the traditional matrix product $C_{\alpha\gamma}$ amounts to a contraction of index β as $C_{\alpha\gamma} = \sum_{\beta=1}^{d} A_{\alpha\beta}B_{\beta\gamma}$, where d represents the different number of values taken by the entries along the dimension indexed by β. In the reliability problems considered in this paper, $d = 2$ so as to capture binary states, although multi-state systems can be considered for values of $d > 2$; however, large d does affect the base of the exponential computational growth with problem dimension size. Despite this, tensors have found successful applications in physics, particularly for representing quantum many-body system states and for computing efficiently on them by coarse-graining problems [20–22]. Although specific developments in physics are beyond the scope of this paper, some recent advances to study strongly correlated systems with non-local interactions in high dimensions, substantiate the capabilities of tensor networks [23]. Their appeal also resides in that the tensor network language is intuitive, as it is a generalization of matrices, which are all familiar to scientists and engineers.

For TNs to offer an alternative perspective in network reliability assessments, this study requires that specialized quantum-inspired tensors be used within a network layout of interest. However, this network should also be consistent with the logic-based *SAT* formulae

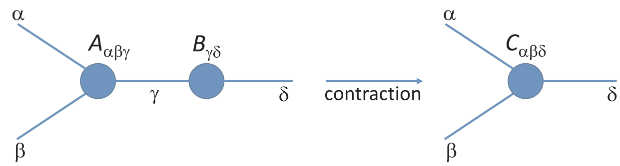

Fig. 1. Tensor network contraction via index γ.

that count satisfiable network configurations (or over-count ATR configurations). Hence, the shared indices across tensors must represent edges of the TN (which are entangled tensors), the gate tensors represent the nodes, and their TN graphical representation shows system connectivity. Note that tensor and connectivity constraints can be handled via tensor contractions (Fig. 1). To establish what the tensors of the network should be to allow for contractions that encode information relevant to engineering reliability, the classical Boolean satisfiability problem *SAT* needs to be invoked. Interestingly, *SAT* on its own continues to show practical success despite its proven computational hardness [24].

1.2. Boolean satisfiability (SAT)

The *SAT* problem relates to the satisfiability of a Boolean formula via an assignment of variables, such that the formula evaluates to 1 or True, after applying the necessary logic operators AND, OR, and NOT, typically represented by $\wedge$, $\vee$, and $\neg$, respectively. A common way of representing Boolean formulas for *SAT* is via their conjunctive normal form (CNF), which highlights the conjunction of clauses, each with a given number of literals or variables [25]:

$$f = \bigwedge_{i=1}^{m} (x_{i1} \vee x_{i2} \cdots \vee x_{ik_i}), \tag{1}$$

where f is the CNF formula, m the number of clauses, and each clause with up to k_i variables, for a total of n distinct variables in the formula, where $n \leqslant \sum_i^m k_i$. Note that this CNF expression is evaluated mathematically via products of sums, which are related to the more canonical sums of products, or disjunctive normal form (DNF), used in engineering to represent reliability polynomials [26]. The CNF formulas can encode the structure of general networks or graphs [27], so it is convenient to imagine one such formula graphically for a simple system composed of link elements in parallel connected in series (Fig. 2), where clauses are nodes and variables are links (using the incidence or bipartite graph representation). The associated SAT_{NET} formula of the graph is $f_1 = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3 \vee x_4 \vee x_5 \vee x_6) \wedge (x_4 \vee x_5 \vee x_6)$ $= (x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6)$ after removing automatically satisfied clauses. This formula with two clauses and six variables can take up to 2^6 variable assignments, of which 49 configurations are satisfiable, such as when all variables equal

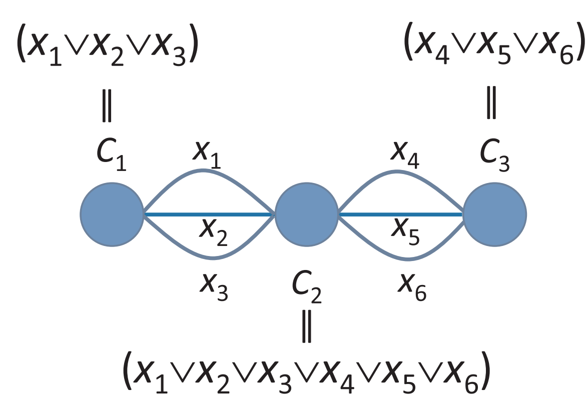

Fig. 2. Network representation of a simple CNF formula with 6 variables (edges) and 3 clauses (nodes).