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A ‘boundary layer’ finite element for thin multi-strake conical shells

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ABSTRACT

Multi-strake cylindrical and conical shells of revolution are complex but commonplace industrial structures which are composed of multiple segments of varying wall thickness. They find application as tanks, silos, circular hollow sections, aerospace structures and wind turbine support towers, amongst others. The modelling of such structures with classical finite elements interpolated using low order polynomial shape functions presents a particular challenge, because many elements must be sacrificed solely in order to accurately represent the regions of local compatibility bending, so-called ‘boundary layers’, near shell boundaries, changes of wall thickness and at other discontinuities. Partitioning schemes must be applied to localise mesh refinement within the boundary layers and avoid excessive model runtimes, a particular concern in incremental nonlinear analyses of large models where matrix systems are handled repeatedly.

In a previous paper, the authors introduced a novel axisymmetric cylindrical shell finite element that was enriched with transcendental shape functions to capture the bending boundary layer exactly, permitting significant economies in the element and degrees of freedom count, mesh design and model generation effort. One element is sufficient per wall strake. This paper extends this work to conical geometries, where axisymmetric elements enriched with Bessel functions accurately capture the bending boundary layer for both ‘shallow’ and ‘steep’ conical strakes, which are characterised by interacting and independent boundary layers, respectively. The bending shape functions are integrated numerically, with several integration schemes investigated for accuracy and efficiency. The potential of the element is illustrated through a stress analysis of a real 22-strake metal wind turbine support tower under self-weight. The work is part of a wider project to design a general three-dimensional ‘boundary layer’ element.

1. Introduction

Cylindrical shells find widespread application as containment structures, supporting structures and aerospace vehicles. Their ubiquity is a result of the relative ease of construction of cylindrical geometries and of the relative simplicity of their manual dimensioning, typically performed using shell membrane theory. This determinate theory is based on balancing external loads with internal membrane stress resultants only, disregarding the high local bending stresses that may arise in response to kinematic compatibility requirements at a boundary or change of wall thickness. These stresses decay away from the discontinuity at an exponential rate, forming a ‘boundary layer’ whose length can be taken as two bending half-wavelengths λ [1]. For a thin cylinder, λ is usually small relative to the length of the strake, and the membrane theory solution is therefore valid over the majority of the cylinder. Where this is not the case, a manual application of axisymmetric shell bending theory is just about practical for uniform thickness cylinders [1–3]. The membrane theory treatment of cones is straightforward due to their straight meridian, however their classical bending theory treatment is made quite challenging by the necessity for the analyst to manipulate Bessel functions [1,4–6]. Cui et al. derived an analytical theory that circumvents the use of Bessel functions while delivering a better accuracy than the equivalent cylinder method [7], but numerical methods tend to be preferred even for stress analyses, although they require a careful mesh design to capture the boundary layer effect.

The authors' previous ‘proof of concept’ study [8] adopted the novel approach of distinguishing between the ‘membrane’ and ‘bending’ components of the shell's kinematic degrees of freedom (DOFs) and interpolating these separately to create a linear axisymmetric ‘Cylindrical Shell Boundary Layer’ (CSBL) element. The membrane displacements were interpolated with simple polynomial functions, but bending displacements were interpolated with transcendental functions derived from the governing differential equation, enriching the element's interpolation field to support the boundary layer natively. An illustration on a number of realistic multi-strake civil engineering shell structures

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showed that the CSBL offered significant advantages in terms of reduced elements and DOFs counts, mesh design and accuracy over a 'classical' shell element with polynomial shape functions based on Zienkiewicz et al. [9].

The same approach will be followed in this paper to derive a conical version of this element, here termed 'CoSBL'. The authors first present a brief derivation of axisymmetric bending theory for conical shells in order to establish the strong form differential equation (following Flügge [4]), as its solution will provide the functional form for the interpolation field of the bending component of the total displacements (the membrane component will be interpolated with simple functions, as for the CSBL). Various integration schemes for the CoSBL stiffness matrix are explored (with results presented in the Appendix for compactness), and two dimensionless parameters are identified to characterise the relationship between the two boundary layers of a CoSBL element. Finally, the potential of the element is illustrated on a complex and realistic 22-strake civil engineering structure.

Readers are invited to consult Chapelle and Bathe [10] for a detailed review of the widespread literature on classical shell finite elements. The authors are aware only of the work of Bhatia and Sekhon that is of direct relevance to this paper, who successfully developed 'macro' cylindrical, conical and spherical linear axisymmetric shell elements [11–13] using a method described in [14]. It does not rely on the definition of bending shape functions, using instead the integration constants of the solution to the governing differential equation as implicit DOFs. The solutions presented accommodate constant distributed loads, although the method supports extension to arbitrary load distributions. Single-strake problems are used for illustration, but the physical significance of the solution and its governing parameters are not discussed in detail.

2. Axisymmetric bending theory for thin isotropic conical shells

The present derivation of the bending theory for isotropic conical shells is adapted from Flügge [4], specialised for axisymmetric cones of constant thickness with all assumptions stated before any equation manipulation. The first step in the derivation, first introduced by Reissner [15], is to solve for the shear force and shell midsurface rotation rather than the radial or meridional displacements. The second step is the identification of the Meissner differential operator [16] allowing for the decoupling of the resulting equations. The last step involves a change of variable from the slant height to a dimensionless parameter to reveal Bessel's differential equation. The physical significance of this parameter and the boundary-layer behaviour of the bending solution is discussed in a later part of the paper.

2.1. Equilibrium, kinematics and constitutive relations

A conical shell of apex half-angle $\pi/2 - \alpha$ (where $0 < \alpha < \pi/2$) and thickness t may be subject to distributed loads p_n and p_s that are respectively normal and tangential to the midsurface (Fig. 1). Assuming axisymmetry of the loading, boundary conditions and geometry, five stress resultants act on the mid-surface: the meridional and circumferential membrane stress resultants n_s and n_θ, the bending moment stress resultants m_s and m_θ, and the meridional transverse shear stress resultant q_s. No displacements, shears or gradients arise in the circumferential θ direction. It is assumed that the conical shell is a frustum bounded by its slant height coordinates s_1 and s_2 ($s_1 < s_2$), leading to the following radial and vertical coordinates:

$$\begin{aligned} r &= s\cdot\cos(\alpha) \\ z &= s\cdot\sin(\alpha)\cdot sgn(\cos(\beta)) + z_0 \end{aligned} \tag{1}$$

Equilibrium considerations yield the following system of equations, where the apostrophe ' denotes differentiation with respect to the slant height s:

$$\begin{aligned} (s\cdot n_s)' - n_\theta &= -s\cdot p_s \\ (s\cdot q_s)' + \tan(\alpha)\cdot n_\theta &= s\cdot p_n \\ (s\cdot m_s)' - m_\theta - s\cdot q_s &= 0 \end{aligned} \tag{2}$$

The following classical linear-elastic constitutive and thin-shell kinematics relationships for a conical shell are adopted (where w and u are the normal and meridional midsurface displacements respectively, while χ is the midsurface rotation about the circumferential axis):

$$\begin{bmatrix} n_s \\ n_\theta \end{bmatrix} = C_m \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_s \\ \varepsilon_\theta \end{bmatrix} \text{ with } C_m = \frac{Et}{(1-\nu^2)}$$
$$\text{and } \begin{bmatrix} m_s \\ m_\theta \end{bmatrix} = C_b \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \kappa_s \\ \kappa_\theta \end{bmatrix} \text{ with } C_b = \frac{Et^3}{12(1-\nu^2)} \tag{3}$$

$$\begin{bmatrix} \varepsilon_z \\ \varepsilon_\theta \end{bmatrix} = \begin{bmatrix} u' & \frac{1}{s}\left(u + \frac{w}{c}\right) \end{bmatrix}^T \text{ with } c = \cot(\alpha)$$
$$\text{and } \begin{bmatrix} \kappa_z \\ \kappa_\theta \end{bmatrix} = \begin{bmatrix} \chi' & \frac{\chi}{s} \end{bmatrix}^T \text{ with } \chi = w' \tag{4}$$

2.2. Uncoupled differential equation

The key to identifying the conical shell bending differential equation is to solve for the variables $s\cdot q_s$ and χ. This requires recasting the membrane kinematic relations as the following:

$$\chi = c((s\varepsilon_\theta)' - \varepsilon_s) \tag{5}$$

From this and the equilibrium equations (Eq. (2)), the following two differential equations are obtained, where the Meissner differential operator Λ can now be identified:

$$\begin{cases} \dfrac{\Lambda(\chi)}{c} = \dfrac{s\cdot q_s}{C_b} \\ \Lambda(s\cdot q_s) + C_m(1-\nu^2)\dfrac{\chi}{c} = g \end{cases}$$
$$\text{where } \begin{cases} \Lambda(f) = c\left[s\cdot f'' + f' - \dfrac{1}{s}f\right] \\ g = -\dfrac{1}{s}\int s(c\cdot p_n + p_s) + c(s^2 p_n)' - \nu\cdot s\cdot p_s \end{cases} \tag{6}$$

A further application of Λ on the second differential equation achieves the decoupling:

$$\Lambda[\Lambda(s\cdot q_s)] + \mu^4 s\cdot q_s = \Lambda(g)$$
$$\text{where } \mu^4 = \frac{12(1-\nu^2)}{t^2} \tag{7}$$

Solutions to this fourth-order real differential equation are the superposition of a particular solution responsible for balancing the loads, referred to as the 'membrane' solution and a linear combination of four functions solution to the homogeneous equation (i.e. for $p_n = p_s = g = 0$), referred to as the 'bending' solution that accommodates boundary conditions. The total value of any quantity is obtained by superposition, e.g. $w = w^b + w^m$ and $n_s = n_s^{\ b} + n_s^{\ m}$.

Once a solution for $s\cdot q_s$ is obtained, the associated stress, strain and displacement fields can be deduced. The second equation from Eq. (6) is used to obtain χ, while the second equilibrium equation in Eq. (2) yields n_θ which, in combination with the first, yields n_s. The bending kinematic relations (Eq. (4)) lead to curvatures which, when combined with the bending constitutive relations (Eq. (3)), are used to obtain m_s and m_θ. The inverse of the membrane constitutive relations (Eq. (3)) can be used to obtain membrane strains from membrane stresses, from which u and w are then finally deduced.