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A unified approach to meshless analysis of thin to moderately thick plates based on a shear-locking-free Mindlin theory formulation

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ABSTRACT

The conventional numerical approximation of Mindlin plate equations can lead to erroneous solutions for thin plates. The so-called shear-locking problem has been well studied in the context of the finite element method (FEM) whereas the development of numerical formulations for its successful elimination in meshfree methods is still a subject of intensive research. This paper studies the effectiveness of some of the most commonly adopted techniques for the reduction of shear-locking and presents the application of a shear-locking-free formulation based on first-order Mindlin plate theory. In this modified formulation, the shear strains are incorporated as degrees of freedom (DOFs) in lieu of the rotational DOFs in the conventional Mindlin theory formulation. A straightforward transformation technique is presented for the enforcement of boundary conditions and comparisons are made with available analytical and numerical solutions. The generalised reproducing kernel particle method (RKPM) is adopted as the numerical tool and a series of numerical examples are presented to demonstrate the accuracy and performance of the presented method.

1. Introduction

The problem of shear-locking in the numerical implementation of Mindlin plate theory is encountered as the plate thickness becomes small and approaches the thin plate limit. Contrary to early expectations, direct applications of the meshfree methods to Mindlin plate problems in the thin plate limit also suffer from the drawback of shear-locking [1]. It has been shown that the cause of this problem is rooted in the inability of the numerical formulation to achieve pure bending states without producing fictitious (spurious) shear deformations [2]. Shear-locking has been extensively studied in the context of the finite element method (FEM) and various strategies have been proposed to alleviate the problem. Reduced integration methods [3,4], assumed strain methods [5,6], mixed interpolation of tensorial components (MITC) methods [7,8] are amongst the techniques that have been widely utilised to eliminate shear-locking. Although being helpful in the context of element-based methods, some of these techniques have shown to be impractical and in some cases unsuitable for meshfree methods [9].

The meshless methods employed on the weak-form formulation can be categorised into two main groups [10], namely the approximant meshless methods [11–16] and the interpolant methods [17–23]. In general, the approximant methods produce smoother variable fields but their shape functions do not fulfil the Kronecker delta property. Consequently, in this group of meshless methods direct enforcement of boundary condition is not possible and special treatments of shape functions or governing equation are required to satisfy the prescribed conditions. The interpolant meshless functions were developed in response to this drawback and are able to produce shape functions that possess the Kronecker delta property [24]. Both types of meshless methods have been employed to Mindlin plate problems in the thin plate limit and in some of these studies the shear-locking phenomenon was encountered. In the meshfree literature, alternative techniques have been proposed to overcome the shear-locking problem. A brief review of the related developments for meshfree methods is presented in the following.

In a similar approach that mimics the reduced integration technique in FEM, nodal integration can be employed to relieve locking. But as shown by Beissel and Belytschko [25], underintegration of the weak form can cause spurious singular modes to occur requiring stabilisation terms to be added to the potential energy functional. In order to avoid shear-locking, Dinis *et al.* [26] adapted the selective integration to the natural neighbour radial point interpolation method (NNRPIM) and showed that it can strongly attenuate the occurrence of shear-locking. Liu *et al.* [27] proposed the conforming radial point interpolation method for static and free vibration analysis of plates, and employed

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strain smoothing stabilisation techniques for nodal integration. Cui *et al.* [28] utilised the cell-based smoothed radial point interpolation method (CE-RPIM) [28] for analysis of shear deformable plates and used only one integration point in each cell to overcome shear-locking. Wang and Chen [2] utilised curvature smoothing stabilisation in the nodally integrated weak form to remedy shear-locking in the limit of thin plate. Cui *et al.* [29] employed the discrete shear gap (DSG) method [30] to mitigate the shear-locking effects. Belinha *et al.* [24,31] analysed thick and composite laminated plates using the natural radial element method (NREM) and employed a "non-centred" integration technique to attenuate shear-locking. Donning and Liu [32] employed the derivatives of the approximation function used for interpolation of translational (DOFs), to approximate rotational DOFs. This approach, which is known as "matching fields" or "unequal order of interpolation", was later extended for the element free Galerkin method (EFGM) by Kanok-Nukulchai *et al.* [33] to eliminate shear-locking in the analysis of beam and plates. Bui *et al.* [34] adopted the matching fields approach for the meshfree buckling analysis of Reissner-Mindlin plates. Recently, Tanaka *et al.* [35] extended the proposed method to the analysis of cracked shear-deformable plates. Tiago and Leitão [9,36] proved that the adaptation of the consistent fields leads to rank deficient system of equations because the constructed approximation functions are linearly dependent. This means that the consistency approach can only be employed if appropriate solvers are utilised that can choose the accurate solution from the set of possible answers.

It is well-known that in the conventional FEM, application of higher-order elements (also known as *p*-refinement) is an effective approach to alleviate the locking effects. This concept has also been explored in the domain of meshfree methods as an effective remedy against locking. Choi and Kim [37] utilised higher-order basis functions in EFGM and studied the optimal order for solutions without shear-locking. Belinha and Dinis [38] employed the EFGM in analysis of plates and laminates, and extensively studied the techniques to avoid shear-locking phenomenon. Garcia *et al.* [39] employed this approach using *hp*-clouds method and demonstrated that the issue of shear-locking can be controlled by using sufficiently high polynomial degrees. Although increasing the degree of the basis functions and constructing higher-order shape functions reduce shear-locking effects, oscillations in the obtained results and locking can occur as the thickness becomes smaller. Cho and Atluri [1] developed a locking-free formulation by changing the considered dependent variables and applied it to Timoshenko beam analysis. In this approach, that is termed as "change of variables", the main idea is to use the strains as primary variables instead of rotation. Tiago and Leitão [9] extended this approach to the analysis of plates and presented the theoretical background for a meshfree locking-free formulation of Mindlin plate theory. This unified approach which can be considered as a modified Mindlin theory has great promise for the analysis of thin to moderately thick plates. Nevertheless, in this formulation direct enforcement of boundary conditions (BCs) along the edges is not possible. This difficulty may be a reason why the modified Mindlin formulation, despite its great potential, has not seen the same interest as other techniques for the elimination of shear-locking in the context of meshfree methods.

This study is conducted using the meshfree generalised reproducing kernel particle method (RKPM) [40–43] which has *p*-refinement capability and allows for the inclusion of any desired order of the derivatives of field variables. The latter feature is favourable in the solution of problems for which a number of BCs involve the derivatives of the field variables. The generalised RKPM was originally proposed by Behzadan *et al.* [43] as a comprehensive form of the gradient RKPM [44–46]. The generalised RKPM and the RKPM combined with finite strip method (RKP-FSM) have been utilised for the bending and buckling analyses of thin plates [24–26] and 3D state-space analysis of thick laminated composite plates [27,28]. In this study, for the first time to the best of authors' knowledge, the modified Mindlin formulation [9] is employed for the numerical analysis of the full range of thin to thick plates. A simple and straightforward numerical transformation is presented to exactly enforce BCs in the modified Mindlin formulation as prescribed in the conventional Mindlin formulations. Moreover, the effectiveness of the *p*-refinement approach in overcoming shear-locking is studied and comparisons are made with analytical and thin plate solutions. To this end, higher- and lower-order interpolations are utilised in the context of the conventional Mindlin theory and the degree of encountered locking for various thickness ratios is thoroughly investigated using a series of benchmark examples. In the presented analysis using Kirchhoff plate theory, the first derivatives of the field variable (deflection) are incorporated in the formulation, thus allowing for the exact and algorithmic enforcement of derivative-type BCs. Without inclusion of slopes as DOFs, recourse to other techniques such as the penalty method and Lagrange multipliers is required to enforce these BCs.

The utilised numerical method is briefly reviewed and the basic governing equation of principle of virtual work is developed into a unified formulation for plate theories. Various plate theories including Kirchhoff, Mindlin, and modified Mindlin theories are presented and discretised by the generalised RKPM. The required procedure for enforcement of BCs in each formulation is presented with required details. Several numerical examples including thick, thin and perforated plates are given to study the effectiveness of the modified Mindlin theory and *p*-refinement approach to alleviate shear-locking. The paper concludes with a summary of accomplishments.

2. Numerical method and general formulation

2.1. The reproducing kernel particle method (RKPM)

A given function $F(\mathbf{x})$ can be estimated over domain ω by the following generalised moving least squares (MLS) approximation [40,43] in terms of the function and its derivatives as

$$\hat{F}(\mathbf{x}) = \sum_{\boldsymbol{\eta}:|\boldsymbol{\eta}|\le k} \int_{\omega} C^{\boldsymbol{\eta}}(\mathbf{x};\mathbf{x}-\boldsymbol{\xi})\ \phi\left(\frac{|\mathbf{x}-\boldsymbol{\xi}|}{\rho(\boldsymbol{\xi})}\right)\ D_{\boldsymbol{\xi}}^{\boldsymbol{\eta}}F(\boldsymbol{\xi})\ \mathrm{d}\boldsymbol{\xi}, \tag{1}$$

in which ϕ is the kernel/window function, $C^{\boldsymbol{\eta}}(\mathbf{x};\mathbf{x}-\boldsymbol{\xi})$ is the associated correction function, and $D_{\boldsymbol{\xi}}^{\boldsymbol{\eta}}F(\boldsymbol{\xi})$ is the $\boldsymbol{\eta}$-th derivative of F with respect to $\boldsymbol{\xi}$. Further, ρ is the dilation parameter, k is the highest order of the derivatives incorporated in the reproduction formula, and $|()|$ is the Euclidian norm of ().

The continuous form of the MLS approximation in Eq. (1) must be discretised using a set of particles to find an approximate solution. Let $\{\mathbf{x}_I\}$ be a set of particles discretising the domain, the formulation of the generalised RKPM is obtained by employing the nodal quadrature rule:

$$\hat{F}_d(\mathbf{x}) = \sum_{I=1}^{NP} \sum_{\boldsymbol{\eta}:|\boldsymbol{\eta}|\le k} C_d^{\boldsymbol{\eta}}(\mathbf{x};\mathbf{x}-\mathbf{x}_I)\ \phi\left(\frac{|\mathbf{x}-\mathbf{x}_I|}{\rho(\mathbf{x}_I)}\right)\ D_{\boldsymbol{\xi}}^{\boldsymbol{\eta}}F(\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}=\mathbf{x}_I}\ \Delta A_I, \tag{2}$$

where NP is the number of particles, ΔA_I is the nodal area associated with I-th particle, and subscript d indicates the discretised formulation. By setting,

$$N_I^{\boldsymbol{\eta}}(\mathbf{x}) = C_d^{\boldsymbol{\eta}}(\mathbf{x};\mathbf{x}-\mathbf{x}_I)\ \phi\left(\frac{|\mathbf{x}-\mathbf{x}_I|}{\rho_I}\right)\Delta A_I, \tag{3}$$

in which $\rho_I = \rho(\mathbf{x}_I)$, Eq. (2) can be rewritten in the following simplified form,

$$\hat{F}_d(\mathbf{x}) = \sum_{I=1}^{NP} \sum_{\boldsymbol{\eta}:|\boldsymbol{\eta}|\le k} N_I^{\boldsymbol{\eta}}(\mathbf{x})\ D_{\boldsymbol{\xi}}^{\boldsymbol{\eta}}F(\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}=\mathbf{x}_I}, \tag{4}$$

where $N_I^{\boldsymbol{\eta}}(\mathbf{x})$ is the shape function of the I-th particle associated with derivative $\boldsymbol{\eta}$ in the generalised RKPM. By setting $k = 0$ where required, the obtained shape functions will be identical to that of the conventional RKPM.