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On the shape of bistable creased strips

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ABSTRACT

We investigate the bistable behaviour of folded thin strips bent along their central crease. Making use of a simple Gauss mapping, we describe the kinematics of a hinge and facet model, which forms a discrete version of the bistable creased strip. The Gauss mapping technique is then generalised for an arbitrary number of hinge lines, which become the generators of a developable surface as the number becomes large. Predictions made for both the discrete model and the creased strip match experimental results well. This study will contribute to the understanding of shell damage mechanisms; bistable creased strips may also be used in novel multistable systems.

1. Introduction

Studies on the deformation of strips with a curved [1] or folded [2] cross-section have only captured smooth deformation of the strip axis. When the cross-section of the strip consists of a sharp crease or fold, new singular bistable behaviour occurs, which is not revealed by previous models. We address this need here.

In their study of bistable creased disks, Lechenault and Adda-Bedia [3] fold a thin disk about equi-spaced radii, in order to create rotationally symmetrical creases. For two or more creases, a central vertex is formed initially, which can be inverted by turning the disk inside out. This configuration is usually bistable, with gently curved material in between creases that have not opened or closed any further. Assuming inextensible *i.e. developable* deformation, and rigidly-fixed crease angles, they calculate the shapes of the initial and inverted states. Their simplest case is a singly creased disk or strip, which can be readily made using, say, paper card of reasonable stiffness, such as a beer mat: after flexing a few times to establish a crease, it can be "pushed through" to form a vertex, where it maintains inversion. We can do the same in Fig. 1 using a plastically folded metal strip. The observed deformation always has two planes of symmetry centred around the central vertex. It is clear the deformed shape shares features with familiar, if unwelcome, indentations straddling fold lines in car body panels, see Fig. 2. This study therefore applies more broadly to folded or pressed thin-walled structures, typically made of metal and used as skins in vehicles, building cladding, *etc*.

Because of the developable assumption, infinite material strains and, hence, stresses and strain energy density occur at the vertex in theory. In practice, these are limited by localised yielding and stretching close to the vertex but not extensively if the disk is relatively thin. This begs an obvious question about how bistability is predicated upon formation of the vertex. As a simple counterexample, consider when a small perforation or hole is made where the vertex would appear, see Fig. 1. The effort needed to push through and invert the strip is now less compared to one of similar size without a hole, and bistability remains, as later experiments attest. There are also no discernible differences in shape, which could be scrutinised, for example, by modifying the analysis in [3] to include a central hole.

We choose instead a *discrete* kinematical formulation in which the deformed shape is approximated by rigid facets folding about hinge lines in the original strip; the central crease is also a hinge line but of fixed rotation. In the limit of a large number of hinges we approach the continuum framework, but this is not essential nor analytically efficient. As shown originally in [4] for the familiar "d-Cone" and then in [5] for general conical defects, the set of compatible rotations for the *least* number of viable hinges have a unique geometrical solution when the vertex is assumed to be developable; the resulting folded shape has remarkably similar overall properties compared to the continuous example it represents. When the apex of the vertex is removed by making a hole, the formulation also applies if we assume that hinge lines intersect at a "virtual" vertex at its centre.

The system is kinematically indeterminate for more hinge lines. If we assume that some form of elastic bending is represented by folding, we may extract, hopefully, a single set of unique rotations that also satisfy equilibrium. We therefore construct the "equivalent" strain energy of bending stored by discrete hinge lines before minimising under the developable vertex constraint. When the vertex hole is reduced in size to zero, the level of strain energy approaches infinity, as noted, just

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Fig. 1. Creased strips with and without a hole in their initial and inverted states. They are made from copper beryllium of thickness 0.1 mm, are 50 mm wide and 200 mm long.

Fig. 2. An inverted strip compared to a “dent” in a vehicle body. Note the similarity of vertex shapes lying on the original crease lines.

as infinite vertex stresses emerge from the continuum analysis in [3], which seem unsatisfactory. However our equilibrium analysis leads to coupled differential equations in the rotations that behave in a bounded manner irrespective of the size of the hole, even if the hole is reduced to zero. In other words, infinite strain energy (and stresses) are largely irrelevant and geometry dominates the nature of solution. This emergent property of the formulation with a hole, we believe, makes an alternative contribution to how such problems are tackled.

The layout is as follows. We first describe the simplest rigid-facet model of a singly-creased strip, where we introduce the Gauss mapping technique for a developable vertex behaviour. We then extend our approach to an arbitrary number of hinge lines, and furnish a worked example to demonstrate a vanishing hole size. Three sets of experimental data are presented and compared to theoretical predictions of the overall shape. When a vertex hole is present, a twofold paradox emerges: the shape of the strip is largely unaffected by the size of the hole even though theory would suggest some differences, and that all specimens appear to conform to the prediction for no hole. We offer a possible explanation but we not do not formally resolve matters before concluding our study. Note that by the nature of our analysis we cannot predict whether or not the inverted configurations are robustly stable but we do give general limits informed by experiments.

2. Kinematic analysis

2.1. Gauss mapping

We construct two phenomenological models of rigid facets connected by straight hinge lines. The deformation is assumed to be doubly symmetric with respect to the crease axis and the transverse centre line of the strip. Non-symmetric deformations may be possible depending on the hinge line geometry, but the bistable behaviour we see always forms a doubly symmetric shape. Consider first a creased strip with a pair of orthogonal hinge lines as shown in Fig. 3a.

When the strip is bent along the crease axis, the crease must flatten completely before rotations about the hinge lines can occur. The second model shown in Fig. 3b has four hinge lines and six facets. It has a second compatible state which is clearly a simplified version of the inverted shape shown in Fig. 1: there is well defined central vertex under a fixed crease angle.

All possible shapes must satisfy rigid folding compatibility of the hinge lines, which can be enforced through a simple Gauss mapping technique [6]. We map the unit normal vector of each facet to the centre of a unit sphere. As the facets rotate, their normal vectors trace out arcs of great circles on the surface of the sphere, with lengths equal to the relative rotation angles between the facets across hinge lines [7]. The signed area enclosed by these arcs is equal to the angular defect at the vertex, which measures the solid angle and, hence, the Gaussian curvature of the vertex [8]. For rigid facets that fold without tearing or crumpling, there can be no defect and thus, the enclosed area must equal zero.

To illustrate how this method works consider the simple two-hinged model shown in Fig. 3a. The facet labelling is shown in Fig. 4a. The rigid facets (A,B,C,D) are labelled according to Bow's notation [9], while the hinge rotation directions satisfy the right-hand rule. The general Gauss mapping is shown in Fig. 4b, which does not correspond to any developable state shown in Fig. 3a. To satisfy the requirement for zero enclosed area, there are two possible cases: $ab = dc = 0$, which is the initial state, or $\beta = 0$, a flattened crease. Since the possible shapes for this configuration of hinges do not correspond to the observed bistable behaviour, more hinges are needed to capture the observed behaviour.

The hinge line layout of Fig. 3b is formalised in Fig. 5. Each of the four hinge lines are symmetrically separated from the crease line by the same angle α, which affords equal rotations, θ, in the deformed case. The six facets (A,B,C,D,E,F) yield a mapping with two crossover points and three enclosed areas, S_1, S_2, S_3, which sum to zero for a developable folded shape. For each area, a right-handed orientation in the sense of following the vectors is declared positive, and *vice versa*. Obviously $ab = bc = de = ef = 0$ gives zero area for $af = cd = \beta$, which is simply the initial strip layout. The second and only non-trivial solution has $ab = bc = de = ef = \theta$ and obviates the following exact relationship between θ, α and β, which is found by calculating the areas using spherical geometry (Appendix A):

$$\tan\frac{\theta}{2} = \frac{\tan(\beta/2)}{\cos\alpha} \tag{1}$$

We are interested, in particular, in the strip end rotation, which is defined as Ψ in Fig. 6 and provides a means of comparison to [3] and experiments. The distance between lines af and cd in the Gauss mapping, which correspond to the crease line segments between facets AF and CD, respectively, is the relative rotation of the crease line segments, or 2Ψ. Therefore, Ψ depends on the facet rotation, θ, but we can eliminate this using Eqn (1) (see Appendix A) in order to return an expression written purely in terms of the fixed parameters, β and α:

$$\sin\frac{\Psi}{2} = \frac{\sin\alpha\,\tan(\beta/2)}{\sqrt{\cos^2\alpha + \tan^2(\beta/2)}} \tag{2}$$

There are no other developable states since other rotations yield a Gauss mapping with net area, which implies stretching of the facets at the vertex as the strip is deformed beyond its initial configuration. As the strip approaches its final developable state and the vertex forms, stretching cannot accumulate and must be relieved. Whilst this leads to possible damage in practice if peak in-plane stresses are high enough, their mitigation by non-linear changes in geometry is a familiar prerequisite for bistable behaviour *e.g.* the snap-through of shallow arch beams [9]. We cannot however prove as such since our model does not capture the intervening deformation; we can only assess the accuracy of Eqn (2) by comparing it to inverted shapes in practice, which we do in Section 3.