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Technical note

On the buckling of an isotropic rectangular plate uniformly compressed on two simply supported edges and with two free unloaded edges

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ABSTRACT

This expository brief note elaborates on the solution to the buckling problem of a uniformly compressed rectangular plate for which the two opposite loaded edges and the unloaded edges are considered simply supported and free, respectively. While Reissner originally discussed briefly this problem in 1909, it was not until 1954 when Ishlinskii presented its rigorous mathematical solution in the form of a transcendental equation, the solution of which requires the use of an iterative numerical technique. Despite remaining almost unreferenced and unnoticed in the literature, the works of Reissner and Ishlinskii in connection with this specific stability problem are educationally instructive and practically relevant. Ishlinskii's solution forms the basis upon which a simple approximate closed-form solution, presented in this note, yields results that do not differ significantly from those computed iteratively using the transcendental equation. Results presented in this brief note can be used as benchmark solutions for validating structural engineering software packages.

1. Introduction

For over a century, solutions of problems related to axially compressed rectangular plates have attracted repeated attention from the structural engineering and mechanics community. This is due to their fundamental roles related to the analysis and design of plated structural components encountered in a wide range of engineering applications. Also, solutions of plate stability problems under various loading and boundary conditions form the fundamental basis upon which local buckling slenderness limits and cross-section classification of metallic structural components are based (e.g. AASHTO [1], ANSI/AISC 360-16 [2], Aluminum Design Manual [3], and Eurocode [4]). The earliest analytical study on the subject can be traced back to the 1890 article by Bryan [5] who presented a solution of the stability problem of an isotropic rectangular plate simply supported on all four edges while subjected to uniform compression on two opposite edges. Since then, a large number of solutions concerning the stability of rectangular plates subjected to various support and loading conditions have been published in classical monographs and archival journals (see e.g. Timoshenko [6], Bleich [7], Timoshenko and Gere [8], Bulson [9], Szilard [10], Chia [11], Bloom and Coffin [12], as well as publications cited in these references). Solutions to the buckling problem of an isotropic rectangular plate compressed on two opposite edges with boundary conditions such that the two loaded edges are simply supported, one unloaded edge is free of any constraints, and one unloaded edge is 1) simply supported, 2) built-in, or 3) rotationally restrained against rotation are well-documented in the literature (Trayer and March [13], Lundquist and Stowell [14], Bleich [7], Bulson [9], Timoshenko and Gere [8]). Surprisingly, the solution to the same problem when both unloaded edges are free has rarely been referenced or brought to the attention of the structural engineering and mechanics community. In 1909 Reissner [15] was the first to discuss this problem and to note, in an elegant mathematical fashion, that the critical force for the problem at hand is attained when the plate buckles into a half- sinusoidal wave in the direction of compression between the loaded edges. Forty-five years following the publication of Reissner's brief note, Ishlinskii [16], in 1954, addressed the same problem and presented a rigorous mathematical solution in the form of a transcendental equation governing the buckling force, the minimum positive real root of which can be used to calculate the critical force. Ishlinskii [16] showed results only for the two limiting cases: a narrow long strip and an infinitely wide plate. He also alerted the reader to the inapplicability of Saint-Venant's principle [17] within the realm of two dimensional plate theory. Enlightening discussions concerning the applicability and limitations of Saint-Venant's Principle to various problems of mechanics and structures can be found in Hoff [18], Mises [19], Sternberg [20], Naghdi [21], Toupin [22], and Gregory and Wan [23].

At a much later date, Banichuk and Ishlinskii [24] discussed again this specific plate problem from both stability and vibration points of view and pointed out that ***an asymmetric buckling*** of the plate with respect to a direction perpendicular to that of the applied forces *does not occur prior to that of a symmetrical configuration*. The objective of this

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technical note is to present a closed-form practical expression for computing the buckling force for an isotropic rectangular plate compressed on two simply supported edges and free of restraints at the unloaded edges. Practicing engineers and engineering students may find the present note not only of instructional value but also of relevance to structural engineering design practice. This note will also serve to provide an attentive response to a question raised via the Steel Interchange of Modern Steel Construction [25] regarding the limiting width-to-thickness ratio one should use for a compressed rectangular steel bar in order to avoid local buckling.

2. Analysis

Consider a homogenous and isotropic rectangular plate, assumed to be perfectly-flat, having width, length, and thickness dimensions of b,l,and t, respectively. Let the plate be subjected to compressive forces N_0, uniformly distributed on two simply supported edges, while keeping the remaining two unloaded edges free of restraints. Hereafter, the plate is assumed to occupy a region of three-dimensional Euclidean space referred to a fixed Cartesian coordinate system (x, y, z) in which the plate middle plane coincides with the (x, y) plane. For convenience, the x-axis is chosen to be the axis of symmetry that is parallel to unloaded edges as shown in Fig. 1.

In this case, the differential equation for the elastic displacement of the plate shown in Fig. 1 can be expressed in the form (see e.g. Timoshenko and Gere [8])

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{N_0}{D}\frac{\partial^2 w}{\partial x^2} \tag{1}$$

where $D = Et^3/12(1-\nu^2)$ in which E and ν are the material's modulus of elasticity and Poisson's ratio, respectively. By adopting a solution (Lévy [26]) of the form $w = f(y)sin(m\pi x/l)$ satisfying the boundary conditions at $x = 0$ and $x = l$, namely $w(0, y) = w(l, y) = 0$ and $M_x(0, y) = M_x(l, y) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = 0$, Eq. (1) can be shown to take the form:

$$\frac{d^4 f(y)}{dy^4} - 2\left(\frac{m\pi}{l}\right)^2\frac{d^2 f(y)}{dy^2} + \left[\left(\frac{m\pi}{l}\right)^4 - \frac{N_0}{D}\left(\frac{m\pi}{l}\right)^2\right] f(y) = 0 \tag{2}$$

Following Ishlinskii [16] and letting $\eta = \frac{\pi y}{l}$ and $\varphi = \frac{N_0}{D}\left(\frac{l}{\pi}\right)^2$, Eq. (2) becomes

$$\frac{d^4 f(\eta)}{d\eta^4} - 2m^2\frac{d^2 f(\eta)}{d\eta^2} + (m^4 - m^2\varphi)f(\eta) = 0 \tag{3}$$

for which the general solution is

$$f(\eta) = C_1\cosh(r\eta) + C_2\cosh(s\eta) + C_3\sinh(r\eta) + C_4\sinh(s\eta) \tag{4}$$

where $r = \sqrt{m^2 + m\sqrt{\varphi}}$ and $s = \sqrt{m^2 - m\sqrt{\varphi}}$; and C_1, C_2, C_3, and C_4 are constants that need be determined from the boundary conditions at the free edges at $y = \pm b/2$ (i.e. both the moment and the shear forces vanish). The moment and shear force equations at the free-edge can be expressed via the following boundary conditions:

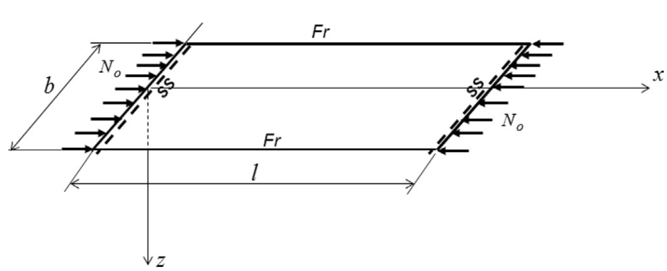

Fig. 1. Coordinate system, rectangular Plate dimensions, loading, and boundary conditions. *SS* and *SFr* denote simply supported and free edges, respectively.

$$M_y\left(x, -\frac{b}{2}\right) = M_y\left(x, \frac{b}{2}\right) = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)\Bigg|_{y=\mp\frac{b}{2}} \tag{5}$$

$$Q_y\left(x, -\frac{b}{2}\right) = Q_y\left(x, \frac{b}{2}\right) = -D\left(\frac{\partial^3 w}{\partial y \partial x^2} + \frac{\partial^3 w}{\partial y^3}\right)\Bigg|_{y=\mp\frac{b}{2}} = 0 \tag{6}$$

By defining, $\beta = (\pi b/2l)$ Eqs. (5) and (6) can be shown to yield:

$$\frac{d^2 f(\eta)}{d\eta^2} - m^2\nu f(\eta)\Bigg|_{\eta=\mp\beta} = 0 \tag{7}$$

$$\frac{d^3 f(\eta)}{d\eta^3} - m^2(2-\nu)\frac{df(\eta)}{d\eta}\Bigg|_{\eta=\mp\beta} = 0 \tag{8}$$

For the case of a symmetrical buckled deflected shape about the x-axis, e.g. $f(y) = f(-y) \equiv f(\eta) = f(-\eta)$, the constants C_3 and C_4 in Eq. (4) must vanish. Thus, $f(\eta)$ becomes

$$f(\eta) = C_1\cosh(r\eta) + C_2\cosh(s\eta) \tag{9}$$

Application of the boundary conditions given in Eqs. (7) and (8) yield the following two simultaneous equations:

$$\begin{bmatrix} r^2\cosh(r\beta) - m^2\nu\cosh(r\beta) & s^2\cosh(s\beta) - m^2\nu\cosh(s\beta) \\ -r^3\sinh(r\beta) + m^2 r(2-\nu)\sinh(r\beta) & -s^3\sinh(s\beta) + m^2 s(2-\nu)\sinh(s\beta) \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10}$$

for which a nontrivial solution exists only if the determinant of the coefficient matrix is equal zero. By doing so and after some algebraic manipulation, the following transcendental equation governing the force parameter φ is obtained:

$$\left[\frac{\sqrt{\varphi} - m(1-\nu)}{\sqrt{\varphi} + m(1-\nu)}\right]^2 = \frac{s\tanh(s\beta)}{r\tanh(r\beta)} = \frac{\sqrt{m^2 - m\sqrt{\varphi}}}{\sqrt{m^2 + m\sqrt{\varphi}}}\frac{\tanh(\sqrt{m^2 - m\sqrt{\varphi}}\,\beta)}{\tanh(\sqrt{m^2 + m\sqrt{\varphi}}\,\beta)} \tag{11}$$

Eq. (11) is exactly the same as that presented by Ishlinskii [16]. This equation must be solved numerically in order to find the minimum positive real root, φ_{min}, thus determining the smallest value of N_0, termed the critical force N_{cr}:

$$N_{cr} = \varphi_{min}\frac{\pi^2 D}{l^2} \tag{12}$$

Substituting the plate flexural rigidity term $D = Et^3/12(1-\nu^2)$ into Eq. (12), the critical buckling force can be computed as

$$P_{cr} = N_{cr}b = \varphi_{min}\frac{\pi^2 Et^3 b}{12(1-\nu^2)l^2} = \frac{\varphi_{min}}{(1-\nu^2)}\frac{\pi^2 EI}{l^2} \tag{13}$$

where $I = bt^3/12$ is the plate moment of inertia about the axis of buckling. Alternatively, the average buckling stress, F_{cr}, can be computed as follows:

$$F_{cr} = \frac{P_{cr}}{A_{cr}} = \frac{\varphi_{min}}{(1-\nu^2)}\frac{\pi^2 E}{l^2}\frac{I}{A_g} = \frac{\varphi_{min}}{(1-\nu^2)}\frac{\pi^2 E}{\left(\frac{l}{r}\right)^2} \tag{14}$$

where $A_g = bt$ and $r = t/\sqrt{12}$ are the plate's gross cross-sectional area and radius of gyration about a centroidal axis parallel to the plate loaded edges, respectively. It is to be noted that Eqs. (12) and (14) differ only slightly (i.e. having the additional term $\varphi_{min}/(1-\nu^2)$) from the classical Euler buckling force and stress formulae for a concentrically compressed elastic member with pin-ended boundary conditions and a length of l.

3. Limiting cases

For the two problems representing the buckling of a narrow plate in one case and the buckling of a very wide plate in another case, closed-form expressions for φ_{min} can be established as follows: