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A new Hamiltonian-based approach for free vibration of a functionally graded orthotropic circular cylindrical shell embedded in an elastic medium

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A B S T R A C T

Exact solutions for free vibration of functionally graded (FG) orthotropic circular cylindrical shells embedded in an elastic medium with arbitrary classical boundary conditions are obtained by a Hamiltonian-based method. Based on the Reissner shell theory and symplectic mathematics, all six possible general solutions instead of trial functions in classical inverse or semi-inverse methods are determined analytically. The determination of natural frequency and vibration mode shape is reduced to an eigen-problem of the Hamiltonian matrix in symplectic space. Numerical examples are provided to verify the validity of the present method. Some new results are also given.

1. Introduction

Shells and shell-like structures have long been key structural components due to their efficient load-carrying capabilities. Recently, with the development of material science, functionally graded materials (FGMs) fabricated by mixing two or more phases of materials for special design requirements [1] have drawn attention of many researchers due to their great promise in applications where the operating conditions are severe [2,3]. The FG circular cylindrical shells are increasingly being used in a variety of engineering applications, such as storage tanks, pressure vessels, submarine pipelines, nuclear pumps and aircraft fuselages [4–11]. Therefore, the study of dynamic behaviors of such shells is of serious consequence for their strength and safety designs considering the wider range of application of FG circular cylindrical shells.

The study on the vibration behaviors of cylindrical shells is a well-established field of research in structural dynamics [12–17]. However, the researches devoted to free vibration of FG circular cylindrical shells are very limited. Vel [18], Cao and Tang [19], Davar et al. [20], Iqbal et al. [21] obtained exact solutions for free vibration of FG cylindrical shells by using some auxiliary and potential functions such as trigonometric functions. Due to the enormous efforts required in obtaining the closed-form solutions, many researchers have resorted to numerical solutions instead of exact solutions. In the framework of numerical methods, the finite element method (FEM) [22–24], meshless method [25,26], Rayleigh-Ritz method [27–29], Galerkin method [30,31] and generalized differential quadrature method (DQM) [32–34] are employed to determine the natural frequency of a FG circular cylindrical shell. Although the numerical data is convenient to aid the design of such structures, it is usually time consuming and enormously costly. Therefore, there are still great demands for analytical methods and exact solutions which can be served as benchmark solutions and benefit the rapid design. Unfortunately, all existing exact solutions were accomplished by inverse or semi-inverse methods involving some pre-determined general solutions (trial functions). No more progress was reported on the topic of closed-form solutions of the FG circular cylindrical shell in recent years.

Motivated by this, the major aim of this paper is to introduce an analytical symplectic method to study the free vibration of a FG orthotropic circular cylindrical shell embedded in an elastic foundation. Unlike the classical analytical method in Lagrangian system, the Hamiltonian-based method [35–38] is systematic and straightforward, and has been successfully extended to the vibration of plates [39–43]. In the Hamiltonian system, the high-order governing differential equation is reduced to a set of ordinary differential equations which can be analytically solved by the method of separation of variables. Six general solutions are analytically determined after obtaining the zero and non-zero symplectic eigenfunctions. Highly accurate natural frequencies and exact vibration mode shape functions for arbitrary combinations of classical boundary conditions are obtained simultaneously. The entire procedure is clear and rational without any pre-determined functions.

The paper is organized in a sequential manner according to theory. The basic equations of a FG circular cylindrical shell are presented in

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Nomenclature

E, υ, ρ	Young's modulus, Poisson's ratio and mass density
L, h, R	Axial length, thickness and middle radius
N	Power-law exponent
ω	Circular frequency
$\mathbf{H}$	Hamiltonian operator matrix
$\boldsymbol{\psi}$	Total unknown vector
$\mathbf{q}, \mathbf{p}$	Original vector and dual vector
$\mu, \boldsymbol{\eta}$	Symplectic eigenvalue and Symplectic eigenfunction
θ_θ	Angle of rotation
A_{ij}, B_{ij}, D_{ij}	Extensional, coupling and bending stiffnesses
K_w	Foundation stiffness
L_C	Lagrangian density function
Q_{kl}	Reduced stiffness
V_x, V_θ	Equivalent shear forces

Section 2 while the Hamiltonian system is established in Section 3. The symplectic eigenvalues, eigenfunctions and exact frequency equations are derived in Section 4. Numerical examples are provided in Section 5. Comparison with existing results whenever possible and benchmark solutions are also included. Finally, Section 6 summarizes with some concluding remarks.

2. Basic equations

Consider a FG orthotropic circular cylindrical shell embedded in an elastic medium having a constant thickness h, an axial length L and a middle radius R as shown in Fig. 1. An orthogonal coordinate system (x, θ, z) is selected at the middle surface of the cylindrical shell where x, θ and z represent the axial, circumferential and radial coordinates, respectively. The displacements in x-, θ - and z- directions are denoted by u, v and w, respectively. The Winkler elastic foundation model is used to describe the reaction of the surrounding elastic medium.

For a FG orthotropic cylindrical shell, the material properties are assumed to vary continuously along the shell thickness [27] and can be represented in the following form of

$$P_x(z) = (P_{xo} - P_{xi})\left(\frac{z + h/2}{h}\right)^N + P_{xi}, \tag{1a}$$

$$P_\theta(z) = (P_{\theta o} - P_{\theta i})\left(\frac{z + h/2}{h}\right)^N + P_{\theta i} \tag{1b}$$

where $P(z)$ are the effective material properties including Young's modulus E, Poisson ratio υ and mass density ρ; the subscripts "x" and "θ" denote x- and θ – axes, respectively; the subscripts "o" and "i" denote the outer and inner surfaces of the cylindrical shell, respectively; N stands for the power-law exponent. In addition, the mass density is assumed as $\rho = \rho_x = \rho_\theta$.

In the thin shell theory, the constitutive relation is given by

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{\theta\theta} \\ \sigma_{x\theta} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{\theta\theta} \\ e_{x\theta} \end{Bmatrix} \tag{2}$$

where σ_{ij} and e_{ij} ($i, j = x, \theta$) are the stress components and strain components, respectively; the reduced stiffness Q_{kl} ($k, l = 1, 2$ and 6) are $Q_{11} = E_x/(1 - \upsilon_x\upsilon_\theta)$, $Q_{22} = E_\theta/(1 - \upsilon_x\upsilon_\theta)$, $Q_{12} = \upsilon_\theta E_x/(1 - \upsilon_x\upsilon_\theta)$, $Q_{21} = \upsilon_x E_\theta/(1 - \upsilon_x\upsilon_\theta)$ and $Q_{66} = E_\theta/[2(1 + \upsilon_x)]$; $\upsilon_x E_\theta = \upsilon_\theta E_x$.

The strain components for a Reissner shell [16] can be further expressed as

$$\begin{Bmatrix} e_{xx} \\ e_{\theta\theta} \\ e_{x\theta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} - \frac{\partial^2 w}{\partial x^2} z \\ \frac{\partial v}{R\partial\theta} + \frac{w}{R} - \frac{1}{R^2}\left(\frac{\partial^2 w}{\partial\theta^2} - \frac{\partial v}{\partial\theta}\right) z \\ \frac{\partial v}{\partial x} + \frac{\partial u}{R\partial\theta} - \frac{1}{R}\left(2\frac{\partial^2 w}{\partial\theta\partial x} - \frac{\partial v}{\partial x}\right) z \end{Bmatrix}. \tag{3}$$

The force and moment resultants of the cylindrical shell are given by

$$\{\{N_x, \ N_\theta, \ N_{x\theta}\}^{\mathrm{T}}, \ \{M_x, \ M_\theta, \ M_{x\theta}\}^{\mathrm{T}}\} = \int_{-h/2}^{h/2} \{\sigma_{xx}, \ \sigma_{\theta\theta}, \ \sigma_{x\theta}\}^{\mathrm{T}}\{1, \ z\} dz. \tag{4}$$

Substituting Eqs. (2) and (3) into Eq. (4), the internal forces can be expressed in terms of mid-surface displacements, i.e.,

$$N_x = A_{11}\frac{\partial u}{\partial x} + \frac{A_{12}}{R}\left(\frac{\partial v}{\partial\theta} + w\right) - B_{11}\frac{\partial^2 w}{\partial x^2} - \frac{B_{12}}{R^2}\left(\frac{\partial^2 w}{\partial\theta^2} - \frac{\partial v}{\partial\theta}\right), \tag{5a}$$

$$N_\theta = A_{21}\frac{\partial u}{\partial x} + \frac{A_{22}}{R}\left(\frac{\partial v}{\partial\theta} + w\right) - B_{21}\frac{\partial^2 w}{\partial x^2} - \frac{B_{22}}{R^2}\left(\frac{\partial^2 w}{\partial\theta^2} - \frac{\partial v}{\partial\theta}\right), \tag{5b}$$

$$N_{x\theta} = A_{66}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{R\partial\theta}\right) - \frac{B_{66}}{R}\left(2\frac{\partial^2 w}{\partial x\partial\theta} - \frac{\partial v}{\partial x}\right), \tag{5c}$$

$$M_x = B_{11}\frac{\partial u}{\partial x} + \frac{B_{12}}{R}\left(\frac{\partial v}{\partial\theta} + w\right) - D_{11}\frac{\partial^2 w}{\partial x^2} - \frac{D_{12}}{R^2}\left(\frac{\partial^2 w}{\partial\theta^2} - \frac{\partial v}{\partial\theta}\right), \tag{5d}$$

$$M_\theta = B_{21}\frac{\partial u}{\partial x} + \frac{B_{22}}{R}\left(\frac{\partial v}{\partial\theta} + w\right) - D_{21}\frac{\partial^2 w}{\partial x^2} - \frac{D_{22}}{R^2}\left(\frac{\partial^2 w}{\partial\theta^2} - \frac{\partial v}{\partial\theta}\right), \tag{5e}$$

$$M_{x\theta} = B_{66}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{R\partial\theta}\right) - \frac{D_{66}}{R}\left(2\frac{\partial^2 w}{\partial x\partial\theta} - \frac{\partial v}{\partial x}\right) \tag{5f}$$

where $\{A_{ij}, \ B_{ij}, \ D_{ij}\} = \int_{-h/2}^{h/2} Q_{ij}\{1, \ z, \ z^2\}dz$ ($i, j = 1, 2$ and 6) are the extensional, coupling and bending stiffnesses which satisfy $A_{12} = A_{21}$, $B_{12} = B_{21}$ and $D_{12} = D_{21}$.

For free harmonic vibration, it is assumed that

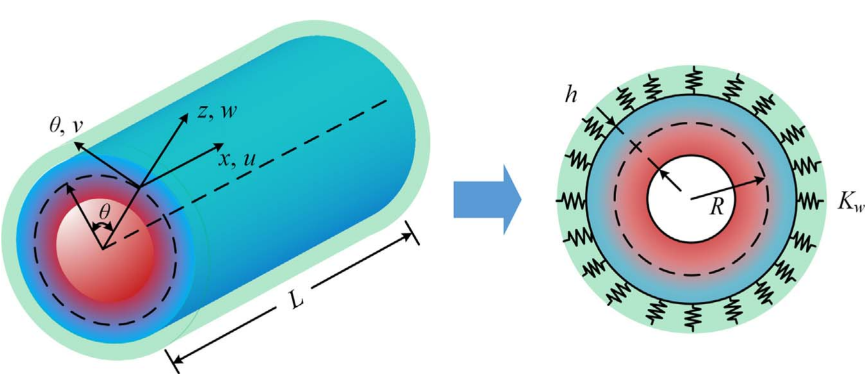

Fig. 1. Geometry of a FG orthotropic circular cylindrical shell embedded in an elastic medium.