Thin-Walled Structures 108 (2016) 381–390

Contents lists available at ScienceDirect

Thin-Walled Structures

journal homepage: www.elsevier.com/locate/tws

Full length article

Vibrations of tori with hollow elliptical cross-section from a three-dimensional theory

Jae-Hoon Kang

Department of Architectural Engineering, Chung-Ang University, 221 Heuksuk-Dong, Dongjak-Ku, Seoul 156-756, South Korea

ARTICLE INFO

Article history:
Received 3 April 2016
Received in revised form
1 September 2016
Accepted 5 September 2016
Available online 14 September 2016

Keywords:
Torus
Toroidal shell
Free vibration
Hollow elliptical cross-section
Legendre polynomial
Ritz method
Three-dimensional analysis

ABSTRACT

Natural frequencies of a toroidal shells of revolution with hollow elliptical cross-section are determined by the Ritz method from a three-dimensional (3-D) theory while traditional shell theories are mathematically two-dimensional (2-D). The Legendre polynomials, which are mathematically orthonomal, are used instead of ordinary algebraic polynomials as admissible functions. The present analysis is based upon the circular cylindrical coordinates while the toroidal coordinates have been used in general. Potential and kinetic energies of the torus are formulated, and upper bound values of the frequencies are obtained by minimizing the frequencies. Convergence to four-digit exactitude is demonstrated for the first five frequencies of the torus. Comparisons are made between the frequencies from the present 3-D method, a 2-D thin shell theory, and thin and thick ring theories. The present method is applicable to very thick toroidal shells as well as thin ones.

1. Introduction

Toroidal shells are elements of many practical science and engineering structures. They are often proposed for rotating spacecraft, neutron accelerators, space colonies, cooling tubes, etc. Unlike straight circular cylindrical shells, the circumferential modes in elastic circular toroidal shells cannot be described by using simple functions. If the circumferential modal (sine or cosine) functions of a straight circular cylindrical shell are employed to describe the circumferential mode shapes of a toroidal shell, they will be strongly coupled making the analysis much more complicated [1]. A vast published literature exists for free vibrations of shells. The monograph of Leissa [2] summarized approximately 1000 relevant publications world-wide through the 1960's. Almost all of these dealt with shells of revolution (e.g., circular cylindrical, conical, spherical). Among them were 14 references considering toroidal shells (see p. 411 in [2]). Some additional investigations of the static and dynamic characteristics of toroidal shells have also been uncovered [1,3–24]. Almost all the research on the toroidal shells has been related to toroidal shells with circular cross-section. Yamada et al. [8] investigated toroidal shells of revolution with hollow elliptical cross-section for the first time. However, their study [8] was based upon conventional thin shell theory. The above mentioned analyses were all based upon experimental methods [3,4] or shell theories which are mathematically two-dimensional (2-D) except for the references [17,18]. That is, for thin shells one assumes the Kirchhoff hypothesis that normals to the shell middle surface remain normal to it during deformations (vibratory, in this case), and unstretched in length. This yields an eighth order set of partial differential equations of motion. For the toroidal shells they involve variable coefficients, making them quite difficult to solve. Even so, conventional shell theory is only applicable to thin shells. Even though a higher order shell theory [19] could be derived which considers the effects of shear deformation and rotary inertia, and would be useful for the low frequency modes of moderately thick shells, such a theory would also be 2-D. But for toroidal shells the resulting equations would be very complicated.

Three-dimensional (3-D) analysis of structural elements has long been a goal of engineers. If one can use 3-D analysis, then the kinematic approximations which are required in 1-D or 2-D representations need not be made (e.g., assuming that plane cross-sections remain plane during deformation of a straight or curved beam, or a ring). With the current availability of computers of increased speed and capacity, it is now possible to perform 3-D structural analyses of bodies in some cases to obtain accurate values of static displacements, free vibration frequencies and mode shapes, and buckling loads and mode shapes. Especially, bodies of revolution permit more efficient 3-D analysis because all mode shapes are Fourier components of the circumferential angle (θ). This allows one to analyze each of the Fourier components separately, and each component entails a problem that has two independent variables in space, instead of three. The first contribution to 3-D vibration analysis of toroidal shells of revolution with hollow circular cross-section was made by Buchanan and Liu [18].

E-mail address: jhkang@cau.ac.kr

http://dx.doi.org/10.1016/j.tws.2016.09.006
0263-8231/© 2016 Elsevier Ltd. All rights reserved.

Nomenclature

a	length of semi-major axis of the middle surface of a hollow elliptical cross-section
a_s	length of semi-major axis of a solid elliptical cross-section
A	cross-sectional area
A_{ij}, B_{kl}, C_{mn}	arbitrary coefficients
b	length of semi-minor axis of the middle surface of a hollow elliptical cross-section
b_s	length of semi-minor axis of a solid elliptical cross-section
D_A	Hillbert space
DET	size of determinant
E	Young's modulus
H	thickness of a hollow elliptical cross-section
H^*	$\equiv H/a$
i, j, k, l, m, n	indices for double summation (non-negative integer)
I	the second moments of cross-sectional area
I_V, I_T	defined by Eqs. (16) and (17), respectively
I,J,K,L,M,N	highest degrees of the Legendre polynomial terms
J	polar moment of cross-sectional area
k	$\equiv b/a$
k_s	shear coefficient used in Timoshenko's beam theory.
K	stiffness matrix
$K_{\alpha\beta\hat{\alpha}\hat{\beta}}$, $M_{\alpha\beta\hat{\alpha}\hat{\beta}}$	submatrix of **K** and **M** ($\alpha = i, k, m$; $\beta = j, l, n$: $\hat{\alpha} = \hat{i}, \hat{k}, \hat{m}$; $\hat{\beta} = \hat{j}, \hat{l}, \hat{n}$)
M	mass matrix
n	circumferential wave number (n=0,1,2,…)
P_n	Legendre polynomial (n=0,1,2,…)
$P_{\alpha\beta}$	$\equiv P_\alpha(\psi)P_\beta(\zeta)$ ($\alpha = i, k, m$, $\beta = j, l, n$)
r	radial coordinate
r_0	radius circle of with the same meridional length of an elliptical cross-section
r,z,θ	circular cylindrical coordinate system
R	distance between the z-axis and the center of a hollow elliptical cross-section
R^*	$\equiv R/a$ or R/a_s
s	mode number
t	time
T	kinetic energy
$T_{\max}$	maximum kinetic energy
TR	total number of the Legendre polynomial terms used in r or Ψ direction
TZ	total number of the Legendre polynomial terms used in z or ζ direction
u_r, u_z, u_θ	displacements in the directions of r, z, θ, respectively
U_r, U_z, U_θ	displacement functions of Ψ and ζ
V	strain energy
$V_{\max}$	maximum strain energy
x	vector of unknown coefficients
z	axial coordinate
$z_{i,o}$	coordinates of the inner and outer surfaces of a hollow elliptical cross-section for $r \geq 0$ and $z \geq 0$, respectively
α	arbitrary phase angle
Γ_1, Γ_2	constants, defined by Eqs. (19).
δ_{ij}	Kronecker delta
ε	$\equiv \varepsilon_{rr} + \varepsilon_{zz} + \varepsilon_{\theta\theta}$
ε_{ij}	tensorial strain
ζ	non-dimensional axial coordinate ($\equiv z/b$ or z/b_s)
$\zeta_{i,o}$	$\equiv z_{i,o}/b$
$\eta_{r,z,\theta}$	functions of ψ and ζ depending upon the geometric boundary conditions
θ	circumferential coordinate
κ_i	functions defined by Eqs. (18) (i=1,2,…,6)
λ, G	Lamé parameters
Λ	domain of a toroidal shell of revolution with a hollow elliptical cross-section
μ	non-dimensional constant of Winkler-Bach curved beam theory
ν	Poisson's ratio
π	3.1415926535…
ρ	mass density per unit volume
σ_{ij}	tensorial stress
ψ	non-dimensional radial coordinate ($\equiv r/a$ or r/a_s)
ψ, ζ, θ	non-dimensional circular cylindrical coordinates
ω	natural frequency
Ω	square of non-dimensional frequency ($\equiv \omega^2 a^2 \rho/G$)
0^A	circumferential wave number for axisymmetric modes
0^T	circumferential wave number for torsional modes
2DS	2-D shell theory
•	time derivative
,	spatial derivative
$[n]$	the largest integer $\leq n$
$\langle f, g\rangle$	$\equiv \iint_\Lambda f(\psi, \zeta)\ g(\psi, \zeta)\ \ \psi\ d\zeta d\psi$

They used a nine-node Lagrangian finite element method based upon the toroidal coordinate system while the present analysis is based upon the circular cylindrical coordinate system. And Zhou et al. [17] analyzed vibrations of torus with solid circular cross-section using the 3-D method for the first time.

In the present study, a 3-D analysis on the vibrations of completely free, toroidal shells of revolution with hollow elliptical cross-section is investigated by the Ritz method. Instead of attempting to solve the equations of motion, an energy approach is followed which, as sufficient freedom is given to the three displacement components, yields frequency values as close to the exact ones as desired. The Legendre polynomials, which are mathematically orthonomal, are used instead of ordinary algebraic polynomials as admissible functions. To evaluate the energy integrations over the toroidal shell volume, displacements and strains are expressed in terms of the circular cylindrical coordinates, instead of related 3-D shell coordinates which are normal and tangent to the shell midsurface, mainly because it takes much more time to compute the energy integration based upon the 3-D shell coordinates than based upon the circular cylindrical coordinates. Comparisons are made between the frequencies from the present 3-D method, a 2-D thin shell theory, and thick and thin ring theories. The present method is applicable to very thick shells as well as thin shells.

2. Method of analysis

A representative cross-section of a toroidal shell of revolution with hollow elliptical cross-section and its planform are shown in Fig. 1. The distance between the axis of revolution (z-axis) and the center of the cross-section is denoted by R. The thickness of the cross-section is H. The lengths of major and minor axes of the midsurface of the elliptical cross-section are $2a$ and $2b$, respectively. The mid-surface of the cross-section has the equation $(r-R)^2/a^2+z^2/b^2=1$ for $r>0$. The circular cylindrical coordinate