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On the tolerance modelling of geometrically nonlinear thin periodic plates

Łukasz Domagalski, Jarosław Jędrysiak*

Department of Structural Mechanics, Łódź University of Technology, al. Politechniki 6, 90-924 Łódź, Poland

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ABSTRACT

There are considered thin periodic plates with moderately large deflections. To take into account the effect of the microstructure on behaviour of these plates the tolerance modelling method is applied, cf. Domagalski and Jędrysiak [2], Meccanica, 2012. This method makes it possible to derive model equations with constant coefficients involving terms dependent of the microstructure size. The paper contains an computational example of critical load calculations and postbuckling analysis.

1. Introduction

Thin linear-elastic plates with a periodic structure in planes parallel to the plate midplane, cf. Fig. 1, are considered. Plates of this kind may undergo deflections of the order of their thickness. Equilibrium problems of such plates are described by nonlinear partial differential equations with non-continuous highly oscillating periodic coefficients. These equations are not suitable in investigating special problems. Therefore, various simplified approaches, introducing effective plate properties, are proposed. Amongst them can be mentioned those based on the asymptotic homogenisation, cf. Kohn and Vogelius [11]. However, the effect of the microstructure size on the plate behaviour in governing equations of these models is usually neglected. Elastostatic problems of thin plates under large deflections are described by the known geometrically nonlinear equations, which are presented in e.g. Timoshenko and Woinowsky-Krieger [17], Woźniak [20]. Using equations of the three-dimensional nonlinear continuum mechanics there are derived equations of von Kármán-type plate theories, Meenen and Altenbach [13]. To investigate bending problems of these plates various methods can be used, for instance the known methods proposed by Levy [12], Timoshenko and Woinowsky-Krieger [17]. Moreover, applications of other new or modified methods are shown in many papers, e.g. a certain asymptotic approach for rectangular plates with variable thickness by Huang Jia-yin [4], a dynamic critical load for buckling of columns by Teter [16], a problem of stability of plates perforated in triangular patterns by Degtyarev and Degtyareva [1], a global and local buckling of sandwich beams and plates by Jasion et al. [5], an analytical study of elastic thin-walled I-section struts buckling using nonlinear variational approach by Wadee and Bai [19].

Some new averaged, non-asymptotic models of thin periodic plates based on the nonlinear theory have been proposed in papers by Domagalski and Jędrysiak [2]. These, so called, tolerance models have been obtained by application of the tolerance averaging technique, cf. the books Woźniak and Wierzbicki [23], Woźniak et al. [21,22]. The obtained equations, in contrary to the exact ones, have constant coefficients, some of which explicitly depend on the characteristic size of a periodicity cell.

The aforementioned technique is very general and is suitable for modelling of problems described by differential equations with highly oscillating coefficients. Since, it is used in analysis of thermomechanical problems of solids and structures with internal microstructure. Applications of this method to various periodic structures are shown in a series of papers, e.g.: for vibrations of periodic wavy-type plates by Michalak [14]; for periodically stiffened plates by Nagórko and Woźniak [15]; for the buckling of periodic thin plates by Jędrysiak [6], where a comparison between critical forces calculated by the tolerance model and by the orthotropic plates theory is shown; for stability analysis of periodic shells by Tomczyk [18]; for dynamic stability of periodic plates by Jędrysiak [7]; for vibrations of periodic thin plates by Jędrysiak [8], where values of fundamental resonance frequencies are compared – for the tolerance and asymptotic models, for the orthotropic plates theory and for the finite element method; for vibrations of thin functionally graded plates with the plate thickness small in comparing to the inhomogeneity

* Corresponding author.
E-mail addresses: lukasz.domagalski@p.lodz.pl (Ł. Domagalski), jarek@p.lodz.pl (J. Jędrysiak).

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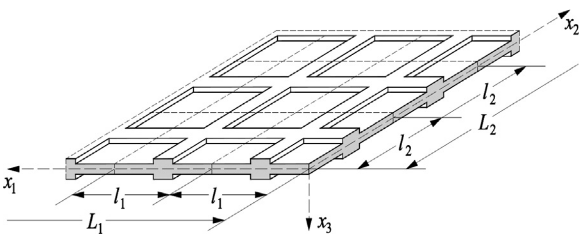

Fig. 1. A fragment of a thin periodic plate.

period by Kaźmierczak and Jędrysiak [10]; and for vibrations of thin functionally graded plates with the inhomogeneity period of an order of the plate thickness by Jędrysiak [9]. The extended list of papers can be found in the books [21,22]. A comparison of preliminary results obtained in the framework of the proposed nonlinear model with finite element solutions was made by Domagalski and Gajdzicki [3].

In this paper the nonlinear tolerance and asymptotic models of elastostatic problems for thin periodic plates with moderately large deflections are presented and discussed. An example of application of these models in analysis of buckling problems of rectangular plates is also shown, with calculation of critical loads.

2. Fundamental equations

Let $Ox_1x_2\times_3$ be an orthogonal Cartesian coordinate system; subscripts i, j, k, l run over 1, 2, 3 and $\alpha, \beta, \gamma, \omega$ run over 1, 2. Denote $\mathbf{x}=(x_1, x_2)$ and $z=x_3$. The undeformed plate with midplane Π and thickness $\delta(\mathbf{x})$ occupies the region $\Omega\equiv\{(\mathbf{x},z)\colon -\delta(\mathbf{x})/2\le z\le\delta(\mathbf{x})/2, \mathbf{x}\in\Pi\}$.

It is assumed that periodic plates under consideration consist of many small repetitive elements called periodicity cells. The cell is defined as a plane region $\square\equiv[-l_1/2, l_1/2]\times[-l_2/2, l_2/2]$, where l_1, l_2 are the cell dimensions along the x_1-, x_2-axis. The size of the microstructure of the plate is described by the diameter of the periodicity cell, given by $l=[(l_1)^2+(l_2)^2]^{1/2}$ and satisfying the condition $\max(\delta)\ll l\ll\min(L_1, L_2)$, ($L_1$ and L_2 are characteristic dimensions of the plate along the x_1- and x_2-axis). This diameter is called *the microstructure parameter*. Hence, the cell can be treated as a thin plate. Let us denote the partial derivatives with respect to a space coordinate by $\partial_\alpha=\partial/\partial x_\alpha$.

Our considerations are based on the nonlinear theory of thin plates [17,20]. Let $w(\mathbf{x})$ be a plate midplane deflection, $u_{0\alpha}(\mathbf{x})$ be the in-plane displacements along the x_α-axes, $F(\mathbf{x})$ be the stress function, and $q(\mathbf{x})$ be the total loadings in the z-axis; $\mathbf{x}\in\Pi$. Thickness $\delta(\mathbf{x})$ can be a periodic function in $\mathbf{x}$ and elastic moduli $a_{ijkl}=a_{ijkl}(\mathbf{x},z)$ can be also periodic functions in $\mathbf{x}$ and even functions in z. Let $a_{\alpha\beta\gamma\delta}$, $a_{\alpha\beta33}$, a_{3333} be the non-zero components of the elastic moduli tensor. Denote $c_{\alpha\beta\gamma\delta}\equiv a_{\alpha\beta\gamma\delta}$-$a_{\alpha\beta33}a_{\gamma\delta33}(a_{3333})^{-1}$.

Define the mean plate properties, being periodic functions in $\mathbf{x}$, i.e. membrane stiffness $b_{\alpha\beta\chi\delta}$, bending stiffness $d_{\alpha\beta\gamma\delta}$, and membrane susceptibility tensor $\tilde{b}_{\alpha\beta\xi\eta}$ components in the form:

$$b_{\alpha\beta\gamma\omega}(\mathbf{x})=\int_{-\delta(\mathbf{x})/2}^{\delta(\mathbf{x})/2}c_{\alpha\beta\gamma\omega}(\mathbf{x},z)dz,\quad d_{\alpha\beta\gamma\omega}(\mathbf{x})=\int_{-\delta(\mathbf{x})/2}^{\delta(\mathbf{x})/2}c_{\alpha\beta\gamma\omega}(\mathbf{x},z)z^2\,dz$$

$$\tilde{b}_{\alpha\beta\xi\eta}b_{\xi\eta\gamma\omega}=\delta_{\alpha\gamma}\delta_{\beta\omega}. \tag{1}$$

In the Föppl-von Kármán formulation of nonlinear theory, substitution of the corresponding derivatives of Airy function $F(\mathbf{x})$ in place of the membrane forces $n_{\alpha\beta}$ is used:

$$n_{\alpha\beta}(\mathbf{x})=\mathfrak{R}_{\alpha\beta}F(\mathbf{x}), \tag{2}$$

where the differential operator $\mathfrak{R}$ is of the form:

$$\mathfrak{R}_{\alpha\beta}(\cdot)\equiv(\delta_{\alpha\beta}\delta_{\xi\eta}-\delta_{\alpha\eta}\delta_{\beta\xi})\partial_{\xi\eta}(\cdot)\equiv(\nabla^2\delta_{\alpha\beta}-\partial_{\beta\alpha})(\cdot), \tag{3}$$

cf. Woźniak [20]. Note that $\mathfrak{R}$ has the following property:

$$\partial_\beta\mathfrak{R}_{\alpha\beta}(\cdot)=\mathfrak{R}_{\alpha\beta}\partial_\beta(\cdot)=0. \tag{4}$$

In the framework of nonlinear two-dimensional thin plate theory the basic relations, involving strain–displacements and membrane strain–membrane forces relations, can be written:

$$\begin{aligned}\kappa_{\alpha\beta}&=-\partial_{\alpha\beta}w,\\ E_{0\alpha\beta}&=\tfrac{1}{2}(\partial_\beta u_{0\alpha}+\partial_\alpha u_{0\beta}+\partial_\alpha w\partial_\beta w),\\ E_{0\alpha\beta}&=\tilde{b}_{\alpha\beta\gamma\omega}\mathfrak{R}_{\gamma\omega}F.\end{aligned} \tag{5}$$

Now, following the modelling procedure presented in the book [21], the action functional can be introduced in the form:

$$\mathcal{A}(w,F,u_{0\alpha})=\int_\Pi\mathcal{L}(\mathbf{x},w,F,u_{0\alpha},\partial_\alpha w,\partial_{\alpha\beta}w,\partial_{\alpha\beta}F,\partial_\beta u_{0\alpha})d\mathbf{x}, \tag{6}$$

$w\equiv w(\mathbf{x})$, $F\equiv F(\mathbf{x})$, $u_{0\alpha}\equiv u_{0\alpha}(\mathbf{x})$, with the lagrangian given as:

$$\begin{aligned}&\mathcal{L}(\mathbf{x},w,F,u_{0\alpha},\partial_\alpha w,\partial_{\alpha\beta}w,\partial_{\alpha\beta}F,\partial_\beta u_{0\alpha})\\ &\quad=\frac{1}{2}(\partial_\beta u_{0\alpha}+\partial_\alpha u_{0\beta}+\partial_\alpha w\partial_\beta w)\mathfrak{R}_{\alpha\beta}F-\tfrac{1}{2}\mathfrak{R}_{\alpha\beta}F\tilde{b}_{\alpha\beta\gamma\omega}\mathfrak{R}_{\gamma\omega}F\\ &\quad\quad+\tfrac{1}{2}\partial_{\alpha\beta}wd_{\alpha\beta\gamma\omega}\partial_{\gamma\omega}w-qw,\end{aligned} \tag{7}$$

where the first two terms correspond to the strain energy due to stretching the middle surface of the plate, the third is related with bending, and the third is the work of external load, all of them averaged over the plate thickness.

One can now formulate the equations of the principle of stationary action,

$$\delta\mathcal{A}(w,F,u_{0\alpha})=0, \tag{8}$$

which, under essential boundary conditions, yield

$$\int_\Pi\left[\left(\mathfrak{R}_{\alpha\beta}\frac{\partial\mathcal{L}}{\partial(\mathfrak{R}_{\alpha\beta}F)}\right)\delta F+\left(\frac{\partial\mathcal{L}}{\partial w}-\partial_\beta\frac{\partial\mathcal{L}}{\partial(\partial_\beta w)}+\partial_{\alpha\beta}\frac{\partial\mathcal{L}}{\partial(\partial_{\alpha\beta}w)}\right)\delta w\right]d\mathbf{x}=0, \tag{9}$$

cf. (4). This equation, together with (7) leads to the following form of the governing equations of nonlinear theory of inhomogeneous anisotropic plates:

$$\begin{aligned}&\mathfrak{R}_{\alpha\beta}(\tilde{b}_{\alpha\beta\gamma\omega}\mathfrak{R}_{\gamma\omega}F-\tfrac{1}{2}\partial_\alpha w\partial_\beta w)=0,\\ &\partial_{\alpha\beta}(d_{\alpha\beta\gamma\omega}\partial_{\gamma\omega}w)-\mathfrak{R}_{\alpha\beta}F\partial_{\alpha\beta}w-q=0,\end{aligned} \tag{10}$$

which will be the starting point of further considerations. In these equations there have been denoted

$$\begin{aligned}\tilde{b}_{\alpha\beta\gamma\omega}&=\tfrac{1}{E\delta}[\tfrac{1+\nu}{2}(\delta_{\alpha\gamma}\delta_{\beta\omega}+\delta_{\alpha\omega}\delta_{\beta\gamma})-\nu\delta_{\alpha\beta}\delta_{\gamma\omega}],\\ d_{\alpha\beta\gamma\omega}&=\frac{E\delta^3}{12(1-\nu^2)}\left[\nu\delta_{\alpha\beta}\delta_{\gamma\omega}+\frac{(1-\nu)}{2}(\delta_{\alpha\gamma}\delta_{\beta\omega}+\delta_{\alpha\omega}\delta_{\beta\gamma})\right],\end{aligned} \tag{11}$$

$E=E(\mathbf{x})$, $\nu=\nu(\mathbf{x})$ stand for the Young's modulus and Poisson's ratio. It can be seen that coefficients of Eq. (10) are discontinuous and highly oscillating. This makes solutions to these equations very difficult to obtain. The main aim of this paper is to propose a replacement of original equations with an approximate model preserving the information about the microstructure of considered plates.

3. Tolerance modelling

3.1. Introductory concepts

Following [21] some of introductory concepts of the tolerance modelling are reminded below.

A cell at $\mathbf{x}\in\Pi_\square$ is denoted by $_\square(\mathbf{x})=\mathbf{x}+_\square$, $\Pi_\square=\{\mathbf{x}\in\Pi\colon {}_\square(\mathbf{x})\subset\Pi\}$. The fundamental concept of the modelling technique is *the*