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# An adaptive model reduction strategy for post-buckling analysis of stiffened structures

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ABSTRACT

The finite element simulation of structures subjected to post-buckling still faces computational limits, especially for large stiffened structures. Several solving strategies have already been proposed in response to this issue. Among them are the adaptive model reduction solving techniques which demonstrated their ability to drastically reduce the number of unknowns as well as to control the approximation error of solving non-linear problems like post-buckling. The challenges regarding these techniques are the computation of a reduced basis at lower cost, the use of an efficient adaptive procedure and the limitation of the number of call to the adaptive procedure. This paper proposes a Post-Buckling Adaptive Model Reduction (PBAMR) strategy, which requires only two initial Ritz vectors without compromising the accuracy of the simulation. This solving method is tested in the case of shear of a stiffened panel.



## 1. Introduction

The stiffened structures used in a variety of applications (especially aeronautics) are likely to buckle locally without compromising their integrity. By allowing this local buckling of structures while preventing permanent deformations, their weight can be further reduced. For this reason, interest in post-buckling has grown recently, leading to the study of induced phenomena, e.g. composite skin/stiffener debonding [1–3]. In these studies or during preliminary and advanced design phases, the post-buckling simulation of structures is needed not only once but several times within optimization procedures [4,5].

Analytical and semi-analytical approaches have demonstrated their relevance in the preliminary design phases through their low computational costs and the establishment of closed-form solutions, which are suited to parametric analysis and optimization [6,7]. However, complex geometries and loadings, material properties and multi-scale phenomena restrict their use in advanced design phases. Although an advanced description of a stiffened panel has been developed by considering elastic restraints due to the stiffeners [7], the buckling interactions between neighbouring panels are not accounted for in these approaches for the moment.

In contrast, the versatility of the finite element method enables it to tackle a wide range of issues. However, the computational cost is much higher and may prevent the simulation of the post-buckling of large scale structures [2,8]. This is the reason why many works have sought to develop efficient solving strategies. Among these are the multi-scale approaches [8–10] and the model reduction techniques [11–15]. In this paper, a projection-based model reduction strategy (i.e. based on the projection of the unknown vector on a Ritz basis) for post-buckling analysis is proposed. Attention is paid to the accuracy of the resulting approximation and to the cost of setting up the post-buckling Ritz basis.

According to aeronautical requirements regarding post-buckling of structures, it is assumed that the targeted applications remain in the framework of the early stage of post-buckling and verify the assumption of small strains and moderated rotations. So the authors focus on a way to use mechanical knowledge of the early stage post-buckling in order to build an efficient Ritz basis, i.e. a small Ritz basis able to represent accurately the solution of the considered problem. In a first section, the state of the art of this knowledge is reviewed in the light of the (semi-)analytical methods. After having recalled the principles of projection-based model reduction and methods to control the approximation error, an adaptive model reduction strategy is proposed. The second section presents the validation of the assumptions and the evaluation of the computational performance of the strategy. The behaviour of the strategy is described in the case of a simple plate. An application to a stiffened panel is then proposed, which is more representative of industrial needs.

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## 2. Literature review

### 2.1. Understanding of the post-buckling behaviour of structures: closed-form solutions

In a first stage of post-buckling, the non-linearity is purely due to geometric effects. This kind of non-linearity is accounted for in mechanical modelling by the assumption of large displacements, small strains and moderate rotations. The non-linear system of equations that governs the static equilibrium of domain $\Omega$ is commonly built from the variational formulation of the potential energy $\Pi_{tot}$ (1) (where $\Pi_{ext}$ is the energy of the external forces):

$$\delta\Pi_{tot}(u) = \int_{\Omega} \underline{\underline{\sigma}} : \delta\underline{\underline{\varepsilon}}(u)\, d\omega - \delta\Pi_{ext} \tag{1}$$

Green–Lagrange strains $\underline{\underline{\varepsilon}}$ (2) and conjugated stresses $\underline{\underline{\sigma}}$ by the Hooke tensor $\underline{\underline{\underline{\underline{C}}}}$ (3) are used in the framework of large displacements, small strains and moderate rotations:

$$\underline{\underline{\varepsilon}} = \tfrac{1}{2}(\nabla u + \nabla^T u) + \nabla u \nabla^T u \tag{2}$$

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}} \tag{3}$$

In the case of plates, the in-plane stresses can also be approximated by the Airy stress functions $\chi(x,y)$ (4), as observed in the literature [7,16]:

$$\underline{\underline{\sigma}} = \begin{pmatrix} \partial_{11}\chi(x,y) & -\partial_{12}\chi(x,y) & 0 \\ -\partial_{12}\chi(x,y) & \partial_{22}\chi(x,y) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

$\partial_{\alpha\beta}g(x,y)$ being the derivative of $g$ according to $\alpha$ and $\beta$, which represent either $x$ or $y$. Approximate expressions of displacement field and/or stress state, that are solutions of equation (1), were sought for simplified configurations by means of analytical Ritz formulations. Besides providing an understanding of the post-buckling, these analytical approximations are based on assumptions that could guide the development of finite element model reduction. Therefore, an insight is given into how a closed-form solution is established for the post-buckling of panels [17,6,18,7].

Generally, the governing equations are derived analytically while the solution is obtained numerically for reasons of non-linearity. The following assumption is taken to hold in the early stage of post-buckling: The post-buckling equilibrium stress/displacement state is close to the fundamental equilibrium state and is obtained by a small variation of the stress/displacement field [17].

Solving the governing equation by the Ritz method has led to various approaches. In the work by Koiter et al. [17] on the post-buckling of a cylindrical panel under axial compression, the variation of the full displacement field due to post-buckling is approximated and the problem is solved with displacement unknowns only. In the case of formulations of flat panel problems [18], the Airy stress function $\chi(x,y)$ and the out-of-plane displacement $w(x, y)$ are the unknowns that respectively enable the in-plane and the out-of-plane variations of stress/displacement fields due to post-buckling to be described. The selected unknowns are approximated by a finite sum of kinematically (respectively statically) admissible displacement (respectively stress) functions.

The accuracy of Ritz methods depends on the number of terms in the sum, as observed by Bisagni et al. [18]. However, increasing the number of terms may reduce the efficiency of the method. In contrast, it is interesting to note that Koiter [19,17], and Vescovini et al. [7] stated the following decomposition of the variation of displacement field due to post-buckling $\delta u_{PB}$:

$$\delta u_{PB} = a \times u_B + \delta\overline{u} \tag{5}$$

where $a$ is the amplitude of the buckling mode $u_B$ and $\delta\overline{u}$ is a small higher order variation. In accordance with this assumption, it was shown by Vescovini et al. [7] that neglecting $\delta\overline{u}$ still led to a small error (less than 7%) in the load–displacement curves. In consequence, the dependence of the accuracy of the approximation on the number of Ritz functions is reduced, as is the total number of unknowns (multiplier factors of the Ritz functions).

Despite the good agreement shown between the maximum values of displacement and stress fields computed by the finite element and semi-analytical methods when neglecting the higher order variation terms [7], it should be noted that their distributions are significantly different. This may have an impact on the design of structures. The higher order variations of displacement field are thus of major importance when it comes to accounting for the stress redistribution in post-buckling. The semi-analytical methods are applied in the range of the early post-buckling stage, which still allows more than twice the critical buckling load to be reached in practice [7,18,17].

Finally, the review of semi-analytical methods highlights the assumptions on stress and displacement fields that have enabled the post-buckling behaviour of plates and shells to be predicted efficiently. The Ritz method is indeed successfully rationalized (two terms are easily computed and can reproduce the solution with relatively good accuracy), which simplifies the user's choice of truncation order of the Ritz functions and reduces the computation cost. From the standpoint of the finite element method, these considerations could be useful to reduce the size of the models. In Section 3, a method is presented which enables the use of *a priori* knowledge from the semi-analytical methods to reduce the finite element model.

### 2.2. Model reduction techniques in non-linear structural finite element analysis

In structural mechanics, the finite element discretization of the governing equations arises from the Galerkin method applied to the variational formulation of the potential energy (1) [20]. The approximation of the displacement field by means of piecewise polynomial interpolation functions results in the matrix form of the static equilibrium at time increment $t_n$ (6):

$$\underline{\underline{K}}(\underline{U}_{|t_n}) \cdot \underline{U}_{|t_n} = \underline{F}_{ext} \tag{6}$$

Due to geometric non-linearities, the stiffness matrix $\underline{\underline{K}}$ depends on the displacement $\underline{U}$. $\underline{F}_{ext}$ is the vector of external forces. Hence a minimization problem can be written, the objective function of which is a norm of residual forces $\underline{R}(\underline{U})$ (Newton–Raphson method), which leads to the iterative solving of the tangent system (7) until a convergence condition is fulfilled. Index $i$ denotes iteration $i$:

$$\underline{\underline{K_t}}(\underline{U}) \cdot \delta\underline{U}^i = -\underline{R}(\underline{U}) \tag{7}$$

The projection-based model reduction techniques have been applied to various non-linear problems [21,22,15,14,11]. By using the *a priori* knowledge of the problems, they aim to reduce the size of the matrix systems presented so as to speed up problem solving. The displacement vector and its increments are basically approximated by a combination of well-chosen Ritz vectors (extracted from the *a priori* knowledge). The vectors form a reduced basis (also called the Ritz basis) $\underline{\underline{C}}$ that spans a subspace $\mathrm{Im}(\underline{\underline{C}})$ of $\mathbb{R}^N$, $N$ being the number of degrees of freedom of the initial FE problem. The approximate displacement and displacement increment are denoted $\underline{U}_C$ and $\delta\underline{U}_C$ and are a linear