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Signed difference analysis: Testing for structure under monotonicity

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HIGHLIGHTS

- Signed difference analysis is re-framed using the theory of oriented matroids.
- State-trace analysis is shown to be a special case of signed difference analysis.
- Additive conjoint measurement shown to be a special case of signed difference analysis.
- A method to fit models in the presence of measurement error is described.

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ABSTRACT

Signed difference analysis (SDA), introduced by Dunn and James (2003), is used to derive testable consequences from a psychological model in which each dependent variable is presumed to be a monotonically increasing function of a linear or nonlinear combination of latent variables. SDA is based on geometric properties of the combination of latent variables that are preserved under arbitrary monotonic transformation and requires estimation neither of these variables nor of the monotonic functions. The aim of the present paper is to connect SDA to the mathematical theory of *oriented matroids*. This serves to situate SDA within an existing formalism, to clarify its conceptual foundation, and to solve outstanding conjectures. We describe the theory of oriented matroids as it applies to SDA and derive tests for both linear and nonlinear models. In addition, we show that *state-trace analysis* is a special case of SDA which we extend to models such as *additive conjoint measurement* where each dependent variable is the same unspecified monotonic function of a linear combination of latent variables. Lastly, we show how measurement error can be accommodated based on the model-fitting approach developed by Kalish et al. (2016).

1. Introduction

Many psychological models can be characterized in terms of functions or mappings between sets of independent, latent, and dependent variables (Dunn & Kalish, 2018). In this characterization, an *input mapping* maps independent variables to a set of theory-dependent latent variables, and an *output mapping* maps the latent variables to a set of dependent variables. Let $\mathbf{y}$ be a vector of the values of n dependent variables, let $\mathbf{x}$ be a vector of the values of m latent variables, and $\mathbf{w}$ be a vector of the values of k independent variables. Then we can write the input mapping as,

$$\mathbf{x} = G(\mathbf{w}),$$

the output mapping as,

$$\mathbf{y} = F(\mathbf{x}),$$

and the full model as,

$$\mathbf{y} = F \circ G(\mathbf{w}).$$

While $\mathbf{y}$ is in principle constrained by both F and G, in many cases, the input mapping is unspecified and interest focuses on testing whether a proposed output mapping can provide an adequate characterization of data. Let $\mathcal{Y} = \{\mathbf{y}_1, \ldots, \mathbf{y}_N\}$ be a set of dependent variable vectors observed under N experimental conditions. Let $P \subseteq \mathbb{R}^m$ be the domain of $\mathbf{x}$. Because $\mathbf{x}$ cannot be measured directly, the experimenter is asking if $\mathcal{Y} \subseteq \{F(\mathbf{x}) : \mathbf{x} \in P\}$. If theory gives an algebraic form for F (e.g., by specifying that F is linear), then it becomes a question of finding a set of latent variable vectors, $\mathcal{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ such that $\mathcal{Y} \approx F(\mathcal{X})$. However, restriction to an algebraic form is often not theoretically reasonable.

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Signed difference analysis (SDA) is concerned with models in which the output mapping F has the form $f \circ g$, where

- $g : P \rightarrow \mathbb{R}^n$ is a *structural mapping*, which maps the set of m latent variables to a set of n *pre-dependent variables* that stand in a one-to-one relationship to the dependent variables of interest.
- $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a *monotonic measurement mapping*. The only assumption on f is that it has the form $f(z_1, \ldots, z_n) = (f_1(z_1), \ldots, f_n(z_n))$, where each f_i is monotonically increasing. The monotonic measurement mapping maps each pre-dependent variable to its corresponding dependent variable.

Models composed of structural and measurement mappings have been proposed in many fields, including judgment and choice (Tversky & Russo, 1969), risky decision making (Kahneman & Tversky, 1979), intertemporal choice (Dai & Busemeyer, 2014), and psychometric testing (Andrich, 1988). Signal detection theory (SDT) also offers examples of this kind of model (Macmillan & Creelman, 2005) which we use to illustrate the distinction. In the most basic SDT model, there are two dependent variables of interest; the *hit rate* (HR) or the probability of reporting a signal when one is present, and the *false alarm rate* (FAR) or the probability of reporting a signal when one is not present. Each dependent variable is modeled as a function of two latent variables; discriminability, d', and a decision criterion, c. That is,

$$\begin{aligned} \mathrm{HR} &= f_1(d' - c) \\ \mathrm{FAR} &= f_2(-c) \end{aligned} \tag{1}$$

where f_1 and f_2 are cumulative distribution functions and therefore monotonically increasing.

It is clear that Eq. (1) can be decomposed into a structural mapping g, and a monotonic measurement mapping f. That is,

$$g : \begin{cases} \mathrm{HR}^* = d' - c \\ \mathrm{FAR}^* = -c \end{cases}$$

where HR^* and FAR^* are pre-dependent latent variables corresponding to the dependent variables, HR and FAR, respectively. And,

$$f : \begin{cases} \mathrm{HR} = f_1(\mathrm{HR}^*) \\ \mathrm{FAR} = f_2(\mathrm{FAR}^*) \end{cases}.$$

SDA tests predictions of a model which do not require estimation of either the values of the latent variables or the form of the monotonic measurement mapping. Instead, predictions are based on geometric properties of the structural mapping that are invariant under all possible monotonically increasing transformations. It is in this sense that SDA tests for 'structure under monotonicity'.

As we discuss in more detail in Section 3.4, SDA can also be viewed as a generalization of *state-trace analysis* (STA), first introduced by Bamber (1979). As in SDA, STA tests a model that can be decomposed into a structural and measurement mapping where each component of the structural mapping is a function of a single latent variable (for further discussion, see Dunn & Kalish, 2018; Dunn, Kalish, & Newell, 2014; Kalish, Dunn, Burdakov, & Sysoev, 2016; Loftus, Oberg, & Dillon, 2004; Newell & Dunn, 2008).

In their initial presentation of SDA, Dunn and James (2003) derived the relevant model predictions from first principles. However, a simpler, clearer, and more productive derivation can be obtained using the mathematical theory of *oriented matroids*. The main aim of the present article is therefore to re-cast the theory of SDA in terms of oriented matroids and, based on this approach, to extend SDA in two directions that were discussed by Dunn and James (2003) without clear resolution. The first is the case in which two or more component functions of the measurement mapping are identical. This is a feature of many models including those derived from SDT as well as *additive conjoint measurement* (ACM) (Krantz, Luce, Suppes, & Tversky, 1971; Luce & Tukey, 1964). The second direction concerns the question of testing a model against data that contain measurement error. We discuss this in light of the procedure recently developed by Kalish et al. (2016) for STA.

The remainder of this article is divided into the following main sections. In Section 2, we outline the aim of SDA in more precise terms. In Section 3, we introduce the theory of oriented matroids in so far as it is relevant to SDA. In Section 4, we use oriented matroids to derive testable consequences for models with a linear structural mapping. In Section 5, we generalize this to a class of nonlinear models with 'near linear' structural mappings. In Section 6, we derive additional testable consequences for linear structural models in which some or all components of the measurement mapping are identical (as in SDT and ACM). In Section 7, we discuss the problem of testing a model in the presence of measurement error and conclude our discussion in Section 8.

2. The aim of SDA

The aim of SDA is to provide a test of a psychological model in which the output mapping consists of a structural mapping and an unspecified but monotonically increasing measurement mapping. Let $\mathbf{y} = (y_1, \ldots, y_n)^T$ be a vector of n dependent variables and let $\mathbf{x} = (x_1, \ldots, x_m)^T$ be a vector of m latent variables. Let $g = (g_1, \ldots, g_n)$ be a structural mapping and let $f = (f_1, \ldots, f_n)$ be a monotonic measurement mapping in which each component function, f_i, is increasing in its argument. Then SDA tests a model where,

$$\mathbf{y} = f \circ g(\mathbf{x}). \tag{2}$$

The logic of SDA rests on the fact that it is possible to identify properties of the structural mapping g that survive arbitrary monotonic transformation. Specifically, the sign of the difference between two vectors of dependent variables observed under different experimental conditions is given by the sign of the difference between two corresponding vectors of pre-dependent variables. The set of such vectors, we show, is determined by geometric properties of g. Therefore, once a model has been defined in terms of its structural mapping, it can in principle be tested using SDA.

Formally, let $Q \subseteq \mathbb{R}^n$ and let $f : Q \rightarrow \mathbb{R}^n$ be a multivariate function of the form $f(x_1, \ldots, x_n) = (f_1(x_1), \ldots, f_n(x_n))$. We say f is *monotonically increasing* (or *monotonic* in short) if each component function, f_i, is strictly increasing in its argument. The *sign* of a vector $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$ is the element of $\{0, +, -\}^n$ whose ith component is the sign of z_i. The sign of a difference between two vectors is called a *signed difference vector*. The key observation is:

Monotonicity Theorem (*Dunn and James, 2003*). *Let $P \subseteq \mathbb{R}^m$, $Q \subseteq \mathbb{R}^n$, $g : P \rightarrow Q$ be a function, and $f : Q \rightarrow \mathbb{R}^n$ be a function. Then if f is monotonically increasing then* $\mathrm{sign}(f \circ g(\mathbf{x}) - f \circ g(\tilde{\mathbf{x}})) = \mathrm{sign}(g(\mathbf{x}) - g(\tilde{\mathbf{x}}))$ *for all* $\mathbf{x}, \tilde{\mathbf{x}} \in P$.

We say that a sign vector X is *observable* under g if there exist vectors, $\mathbf{x}, \tilde{\mathbf{x}} \in P$, such that $\mathrm{sign}(g(\mathbf{x}) - g(\tilde{\mathbf{x}})) = X$. It follows from the Monotonicity Theorem that if X is observable under g it is also observable under $f \circ g$ for any monotonically increasing measurement mapping, f. In Section 4 we will extend this theorem to include a *weakly* monotonic measurement mapping in which each component function is strictly non-decreasing.

By way of example, consider a model $\mathbf{y} = f \circ g(\mathbf{x})$, for $\mathbf{y} = [y_1, y_2]$, f monotonic and $g(x) = (ax + c, bx + d)$, for constants, $a, b > 0$, c, and d. Thus the image of the function g is a line with positive slope. Without specifying f, all we can say about possible values of $\mathbf{y}$ is that they should lie on a curve $\{y_2 = h(y_1)\}$ which is the graph of an increasing function h. Equivalently, any two