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Quantum like modeling of decision making: Quantifying uncertainty with the aid of Heisenberg–Robertson inequality

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HIGHLIGHTS

- Uncertainty in the process of decision making is quantified with the aid of the Heisenberg–Robertson inequality.
- This approach demonstrates dependence of incompatibility of questions on the mental state of decision maker.
- A one parametric family of operators representing incompatible questions as a continuous deformation of compatible is constructed and this formalism is applied to modeling of the order effect statistical data.

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ABSTRACT

This paper contributes to quantum-like modeling of decision making (DM) under uncertainty through application of Heisenberg's uncertainty principle (in the form of the Robertson inequality). In this paper we apply this instrument to quantify uncertainty in DM performed by quantum-like agents. As an example, we apply the Heisenberg uncertainty principle to the determination of mutual interrelation of uncertainties for "incompatible questions" used to be asked in political opinion pools. We also consider the problem of representation of decision problems, e.g., in the form of questions, by Hermitian operators, commuting and noncommuting, corresponding to compatible and incompatible questions respectively. Our construction unifies the two different situations (compatible versus incompatible mental observables), by means of a single Hilbert space and of a deformation parameter which can be tuned to describe these opposite cases. One of the main foundational consequences of this paper for cognitive psychology is formalization of the mutual uncertainty about incompatible questions with the aid of Heisenberg's uncertainty principle implying the mental state dependence of (in)compatibility of questions.

1. Introduction

During the recent years the quantum-like approach to modeling of cognition and decision making (DM) under uncertainty has been increasingly applied to behavioral results surprising or problematic from classical perspectives.[1] One of the main distinguishing features of this approach is the possibility to treat mutually incompatible ("complementary") DM problems, e.g., questions, inside the common model based on quantum probability. Experts in "classical DM-theory" were well aware about the existence of such problems, e.g., in the form of the disjunction, conjunction, and order effects (see, e.g., Tversky & Shafir, 1992). The attempts to represent incompatible problems in the classical

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[1] As some representative works, we can mention the following: Aerts, Broekaert, Gabora, and Sozzo (2016), Asano, Basieva, Khrennikov, Ohya, and Tanaka (2012, 2017), Asano, Khrennikov, Ohya, Tanaka, and Yamato (2015), Bagarello (2012, 2015), Bagarello, Di Salvo, Gargano, and Oliveri (2017), Boyer-Kassem, Duchene, and Guerci (2016), Busemeyer and Bruza (2012), Busemeyer, Pothos, Franco, and Trueblood (2011), Busemeyer, Wang, and Townsend (2006), de Barros (2012), de Barros and Oas (2014), Dzhafarov, Cervantes, and Kujala (2017), Dzhafarov and Kujala (2014a, 2014b, 2016, 2017), Haven and Khrennikov (2013, 2016), Haven and Sozzo (2015), Khrennikov (2003, 2004a, b, 2010, 2016), Khrennikov and Basieva (2014), Khrennikov and Haven (2007), Khrennikova and Haven (2016), Plotnitsky (2014), Pothos and Busemeyer (2009, 2013), Pothos, Perry, Corr, Matthew, and Busemeyer (2011), and Trueblood and Busemeyer (2012), Wang and Busemeyer (2013), Zhang and Dzhafarov (2015), and references therein.

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probabilistic framework led to a number of paradoxes and theoretical proposals augmenting classical probabilistic inference with additional assumptions (e.g., Costello & Watts, 2014; Tentori, Crupi, & Russo, 2013). The best known are the Allais (1953), Ellsberg (1961) and Machina (1982) paradoxes, but in their review Erev, Ert, Plonsky, Cohen, and Cohen (2016) count 35 basic paradoxes of classical DM-theory.

Quantum-like modeling of DM, or more generally, cognition is based on the quantum methodology and formalism, but not on quantum biophysics (cf., e.g., works of Hameroff (1994) and Penrose (1989) about reduction of cognition to quantum physical processes in the brain). In the quantum-like framework the brain is a black box, such that its information processing can be described by the formalism of quantum theory. "Mental observables", e.g., in the form of questions, are represented by Hermitian operators (and in more general framework by so-called positive operator valued measures, Asano et al., 2015). The mental state (or the belief state) of an agent is represented like a quantum state, i.e., a normalized vector of the state space (or, more generally, a density operator representing the classical statistical mixture of pure states).

Therefore we can apply the *Heisenberg uncertainty principle* to characterize interrelation of uncertainties of two incompatible questions (or tasks) A and B. In the general form the Heisenberg uncertainty principle is expressed in the form of the *Robertson inequality:*

$$\sigma_A(\psi)\sigma_B(\psi) \geq |\langle [A, B]/2\rangle_\psi|, \tag{1.1}$$

where $[A, B] = AB - BA$ is the commutator of the operators, $\langle [A, B]\rangle_\psi$ is the mean value of the commutator with respect to the state ψ, $\sigma_A(\psi), \sigma_B(\psi)$ are the standard deviations of the observables A and B with respect to the state ψ.

The operators representing the position and momentum observables satisfy a very special commutation relation (the canonical commutation relation): $[q, p] = i\mathbb{1}$, where $\mathbb{1}$ is the unit operator. By using this relation and the Robertson inequality we obtain the original Heisenberg inequality:

$$\sigma_q(\psi)\sigma_p(\psi) \geq 1/2. \tag{1.2}$$

We emphasize that the latter imposes a state-independent constraint onto the product of standard deviations, since the right-hand side of (1.2) does not depend on the state ψ. This is very important property of the Heisenberg inequality. In general we do not have a state independent estimate of the form $\sigma_q(\psi)\sigma_p(\psi) \geq c$, where $c > 0$ does not depend on ψ (cf. with (1.2)). The lower bound for the *interrelation between the standard deviations is state dependent.*

Thus, even for noncommuting mental observables A and B, the right-hand side of the Robertson inequality (1.1) can be equal to zero. In this case the observables A and B are similar to classical observables. In particular, if $[A, B]\psi = 0$ for some mental state, we assume an equivalence with classical probability description in the form of random variables, see Section 3. In quantum foundations this issue was studied in very detail by Ozawa (2006, 2011, 2016) and we shall apply his approach to DM and cognition, Section 3. In that section we shall refer to the condition of *spectral commutativity*. The latter is equivalent to condition $[A, B]\psi = 0$ in the case of dichotomous observables (which we focus on in this paper). However, for general observables $[A, B]\psi = 0$ does not imply spectral commutativity and hence does not imply the possibility of using the classical probability model.[2] (Note that the condition $[A^n, B^m]\psi = 0$ for any n, m is equivalent to the spectral commutativity of A, B.)

The more general situation, $\langle [A, B]\rangle_\psi = 0$, is more complicated from the interpretational viewpoint (Section 3). We note that in the finite-dimensional space (used for representation of beliefs) it is impossible to construct Hermitian operators satisfying the canonical commutation relation. Moreover, any Hermitian operator has eigenvectors and, for states consistent with them, variance equals zero and (1.1) degenerates to $0 \geq 0$.

The state dependence of the uncertainty relations for mental observables was emphasized by Khrennikov and Haven (2007). The role of the principle of complementarity in cognitive science was analyzed by Khrennikov (1999) and Wang and Busemeyer (2013).

Section 2 contains the basic mathematical construction that unifies the two different situations (compatible versus incompatible mental observables) by means of a single Hilbert space and a deformation parameter θ that can be tuned to describe these opposite cases (cf. the work of Pothos and Busemeyer (2013), where these cases were treated separately and in different state spaces). The one-parametric families of operators can be used for quantum-like modeling in cognitive psychology and psychophysics — by treating θ as the formal parameter (representing the degree of deformation of compatibility) and selecting it to match experimental data. In Section 5 we use this approach to construct the Hermitian operator representation of questions demonstrating the order effect: we match the deformation parameter θ with the degree of noncommutativity in the sequential joint probability distributions obtained on the basis of the experimental data taken from Moore (2002). These operators can be used in the quantum-like model of Busemeyer and Pothos (2013). Section 3 presents the most important (for psychological applications) message of this paper: the state dependence of incompatibility of questions. Thus to be or not to be compatible depends not only on questions, but also on the mental state. This statement is very natural from the cognitive viewpoint and our contribution is to put it into the formal mathematical framework.

2. Operator representation of incompatible and compatible questions ("mental observables")

We work in finite-dimensional (complex) Hilbert spaces. Such space $\mathcal{H}$ can be represented (by fixing an orthonormal basis) as the space of vectors $\psi = (\psi_1, \ldots, \psi_n)$ with complex coordinates, endowed with the scalar product given as $\langle \psi, \phi\rangle = \sum_i \psi_i \bar{\phi}_i$. *Mental states* are represented by normalized vectors of $\mathcal{H}$, and mental observables, e.g., in the form of questions, are represented by Hermitian operators.

Consider some decision maker, call her Alice. Following Busemeyer and Pothos (2013), we consider the following pair of aspects of Alice's life represented in the form of questions (mental observables):

- Q_1: "Are you happy or not?"
- Q_2: "Are you employed or not?"

We represent each aspect of Alice's life in its own Hilbert state space. The *happiness status* is modeled as a two-state system living in the two-dimensional Hilbert space $\mathcal{H}_H = \mathbb{C}^2$. We introduce an orthonormal basis $\mathcal{F}_H = \{h_+, h_-\}$ of $\mathcal{H}_H$, and a Hermitian operator H, the *happiness operator*, having $h_\pm$ as eigenstates: $Hh_\pm = \pm h_\pm$. Of course, we have $\langle h_j, h_k\rangle = \delta_{jk}, j, k = \pm$. The interpretation of eigenstates of the happiness operator is clear: if Alice's state is $\Psi = h_+$, then she is definitively happy. But she is unhappy if $\Psi = h_-$. The crucial point is that the state of happiness is not always explicitly determined; Alice can be in the state of superposition of happiness and unhappiness. Such a mental state is represented by a linear combination $\Psi = \alpha_+ h_+ + \alpha_- h_-$, with $|\alpha_+|^2 + |\alpha_-|^2 = 1$. In

[2] Here we speak about the *noncontextual* classical probability model. Contextual classical measure-theoretic models can serve even for representation of incompatible observables (see Khrennikov, 2010, and Dzhafarov & Kujala, 2014a; 2014b; 2016; 2017; Dzhafarov et al., 2017).