Journal of Mathematical Psychology 82 (2018) 52–55

Contents lists available at ScienceDirect

Journal of Mathematical Psychology

journal homepage: www.elsevier.com/locate/jmp

Complementary symmetry in Cumulative Prospect Theory with random reference

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HIGHLIGHTS

- We study Cumulative Prospect Theory models of buying and selling prices.
- In Model 1 the gramble's prizes are integrated with the price.
- In Model 1 complementary symmetry holds for any kind of loss or risk attitude.
- In Model 2 the utility of a gamble is balanced against the price.
- Constant buying/selling price ratio in Model 2 relies on preference homogeneity.

ARTICLE INFO

Article history:
Received 22 February 2017
Received in revised form 4 November 2017

Keywords:
WTA
WTP
Reference dependence
Third-generation prospect theory
Preference homogeneity
Loss aversion
Reflection

ABSTRACT

It is shown that complementary symmetry holds in Cumulative Prospect Theory with random reference if the utility function for gains and losses is strictly increasing and continuous. Previous results imposed more restrictions involving preference homogeneity, reflection, and loss aversion. The result holds true in the general version of the Third-Generation Prospect Theory provided that the relative value function v takes the same form as in Cumulative Prospect Theory.

1. Introduction

Complementary symmetry is a property introduced by Birnbaum and Zimmermann (1998). It involves two binary gambles $g := (x, p; y)$ and $g' := (x, 1-p; y)$, where x, y are two monetary outcomes such that $x > y$, and $p \in (0, 1)$ is the probability of receiving x in g and y in g'. It says that the sum of the buying price of g – i.e. the largest amount an individual is willing to pay for g, denoted by $b(g)$ – and the selling price of g' – i.e. the smallest amount an individual is willing to accept to forfeit g', denoted by $s(g')$ – equals the sum of the two monetary outcomes: $x + y$.

This property has been shown to fail in experimental settings (Birnbaum and Sutton, 1992; Birnbaum, Yeary, Luce, and Zhao, 2016, and Birnbaum and Zimmermann, 1998). The experiments were designed to elicit buying and selling prices for each individual in a group of subjects for a series of binary gambles g, g' where the amount $x + y$ was held constant. It was found that the sum of the median $b(g)$ value and the median $s(g')$ value is not constant and depends on the range $x-y$. The sum is always below the value $x+y$ and decreases as the range becomes wider. For example, Birnbaum and Sutton (1992) show that the median buying and selling prices of (\$60, 0.5; \$48) are \$50 and \$54, respectively, and thus their sum equals \$104. However, the median buying and selling prices of (\$96, 0.5; \$12) are \$25 and \$50, respectively, and their sum equals a meager \$75.

In the buying/selling price elicitation task the decision maker is asked to make a trade-off between the gamble in question and a sure amount of money that the gamble is exchanged for. The seller exchanges the gamble he owns for a sure amount of money whereas the buyer exchanges the money in his possession for the gamble he wants to acquire. In order to model this kind of asymmetric trade-off within Cumulative Prospect Theory (CPT in short, Tversky and Kahneman, 1992) Birnbaum and Zimmermann (1998) (Appendix B) proposed two models. In model 1 the utility of a gamble is balanced against the price (obtained when selling

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https://doi.org/10.1016/j.jmp.2017.11.004

or paid when buying). This model is an extension of the model of Tversky and Kahneman (1991) that was proposed for goods to gambles. In Model 2 the gamble's monetary prizes are integrated with the price: the price serves as reference to evaluate the gamble when buying and the gamble serves as (random) reference to evaluate the price when selling. Birnbaum and Zimmermann (1998) identified the key implications of each of the two models and have shown that they are inconsistent with the experimental evidence suggesting that the range of outcomes, i.e. $|x-y|$, plays an important role in the price elicitation tasks (see for instance Birnbaum and Beeghley, 1997; Birnbaum and Stegner, 1979; Birnbaum and Sutton, 1992). In the case of Model 2 the questionable implication identified by Birnbaum and Zimmermann (1998) is complementary symmetry, whereas in the case of Model 1 it is constant selling to buying price ratio.

The purpose of this note is to show whether these implications carry over to the case where we relax some of the strong assumptions of the parametric CPT model that were used in Birnbaum and Zimmermann (1998). We shall mainly focus on Model 2 (and hence on complementary symmetry) because the main idea of this model, i.e. the integration of prizes and prices, has become standard in later accounts (e.g. Luce, 1991) and, in particular, has been adopted in the Third-Generation Prospect Theory (PT^3 in short, Schmidt, Starmer, and Sugden, 2008). We study the less popular Model 1 and its implication of constant selling to buying price ratio in Appendix.

Within model 2 Birnbaum et al. (2016) and Birnbaum and Zimmermann (1998) were able to demonstrate that, irrespective of the form of the probability weighting functions for gains and losses, complementary symmetry holds if utility function for gains and losses is of the following form:

$$u(x)=\begin{cases} x^{\alpha}, & \text{for } x\geq 0,\\ -\lambda(-x)^{\alpha}, & \text{for } x<0.\end{cases} \quad \text{where } \alpha>0. \tag{1}$$

The main contribution of this note is to show that complementary symmetry holds more generally in this model for any strictly increasing and continuous utility function satisfying $u(0)=0$. In particular, it holds irrespective of whether any kind of loss aversion, or reflection,[1] or preference homogeneity (power utility) is assumed. Section 2 introduces the model and proves the main result. Section 3 shows how the results are carried over to the more general PT^3 model. Appendix analyses the implications of Model 1 of Birnbaum and Zimmermann (1998).

2. Complementary symmetry in cumulative prospect theory with random reference

Except for a few exceptions, we adopt the same setup and notation as in Birnbaum et al. (2016) to enhance comparability. Let $(x,p;y)$ be a binary prospect, in which the outcome is x with probability $p\in(0,1)$ and y otherwise, where $x,y\in\mathbf{R}$ and $x>y$. It is assumed that outcomes x and y are defined relative to some reference outcome that is normalized to zero; a negative outcome is thus perceived as a loss and a positive one as a gain. The CPT model for $(x,p;y)$ is then written as follows:

$$U(x,p;y) = \begin{cases} u(x)w^{+}(p)+u(y)[1-w^{+}(p)], & \text{if } x>y\geq 0,\\ u(x)w^{+}(p)+u(y)w^{-}(1-p), & \text{if } x\geq 0\geq y,\\ u(x)[1-w^{-}(1-p)]+u(y)w^{-}(1-p), & \text{if } 0\geq x>y \end{cases} \tag{2}$$

[1] It can be proved for CPT that changing the sign of all consequences of two prospects always reverses the preference is equivalent to assuming the (strictly increasing and continuous) utility function of the following form: $u(0)=0$, $u(x)=-\lambda u(-x)$, for all $x>0$, where $\lambda>0$.

where $u:\mathbf{R}\to\mathbf{R}$ is a strictly increasing and continuous utility (value) function satisfying $u(0)=0$, and $w^{+}:[0,1]\to[0,1]$, $w^{-}:[0,1]\to[0,1]$ are strictly increasing and continuous probability weighting functions for gains and losses, respectively, satisfying $w^{+}(0)=w^{-}(0)=0$ and $w^{+}(1)=w^{-}(1)=1$.

A gamble is the same as a prospect except that a former may not be defined relative to a reference outcome whereas the latter always is. In the CPT model we subtract a reference outcome from each outcome of a gamble to form a prospect. In what follows we allow the possibility of a random reference outcome. Generally, this would require the state-space approach such as in the PT^3 model in order to take into account the dependence structure between an evaluated gamble and a reference gamble. However, when modeling the buying and the selling prices, it is never the case that a reference and an evaluated object are both random.[2] Hence, to keep things simple, we will stick to the simple setup of gambles being probability distributions. In the non-standard case of evaluating a sure outcome relative to a random reference, we shall form a prospect by subtracting each outcome of the gamble from the sure alternative in each state.

Consider a gamble $g:=(x,p;y)$, where $x>y$. We define the buying price $b(g)\in\mathbf{R}$ as the maximal sure outcome for which the decision maker is willing to buy g when it is evaluated relative to the reference outcome $b(g)$. Similarly we define the selling price $s(g)\in\mathbf{R}$ as a minimal sure outcome for which the decision maker is willing to sell g when $s(g)$ is evaluated relative to g. The prices thus satisfy the following conditions:

$$U[x-b(g),p;y-b(g)]=0, \tag{3}$$

$$U[s(g)-x,p;s(g)-y]=0. \tag{4}$$

Having defined the model we now state the main result.

Proposition 2.1. *Consider two gambles $g:=(x,p;y)$ and $g':=(x,1-p;y)$, where $x>y$, and $p\in(0,1)$. In the model defined by (2), (3) and (4) the following holds:*

$$b(g)+s(g')=x+y. \tag{5}$$

Proof. To save on notation denote $s:=s(g')$ and $b:=b(g)$. Note first that by (2), monotonicity of u and the fact that $u(0)=0$, it must be that both prices lie between the lower and the upper outcome of the corresponding prospect, i.e. $b,s\in(y,x)$. Hence they satisfy:

$$w^{+}(p)u(x-b)+w^{-}(1-p)u(y-b)=0$$

$$w^{-}(1-p)u(s-x)+w^{+}(p)u(s-y)=0.$$

Or after rearranging and combining:

$$\frac{u(s-y)}{-u(s-x)}=\frac{u(x-b)}{-u(y-b)}=\theta(p), \tag{6}$$

where $\theta(p):=\frac{w^{-}(1-p)}{w^{+}(p)}$. Suppose now that contrary to the claim it is not true that $b+s=x+y$. There are two cases to consider:

[2] One can argue, however, that the buying or the selling price is not a point-estimate but either a random variable or a fuzzy number. The intuition behind is that it is often difficult to choose a crisp numerical value below which the decision maker will not sell (or above which she will not buy). It may be that there is an interval of prices $[s_l,s_u]$, $s_l<s_u$ such that the decision maker is sure that she would not sell below s_l and is sure she would sell above s_u, and she remains hesitant in between s_l and s_u. While we believe that this is a valid possibility, we decided not to follow this path, as it would require a different preference structure, allowing for instance the violations of completeness or of transitivity of indifference.