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A generalized extensive structure that is equipped with a right action and its representation

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HIGHLIGHTS

- The set of commodities (the base set) is a generalized extensive structure.
- Decomposability is supposed for the product of the base set and a set of durations.
- Both ordering and algebraic axioms are proposed for the decomposable structure.
- Properties derived under the axioms yield right action of duration on the base set.
- We get a weighted additive model so as to reflect nonconstant impatience.

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ABSTRACT

In intertemporal choice, it has been found that if the receipt time is closer to the present, then people tend to grow increasingly or decreasingly impatient. This paper develops an axiom system to construct a weighted additive model reflecting nonconstant impatience. By presupposing that an increment in duration is subjectively assessed according to the periods at which advancement occurs, we denote the one-period advanced receipt of outcomes by multiplying the outcomes by the increment on the right. By this right multiplication, we can regard the effect of advance as the decomposition into two factors, i.e., the factor of step-by-step advance accompanied by subdivided durations and the factor of advance based on the total duration. First, the conditions for enabling right multiplication are proposed for the Cartesian product of the underlying set of a generalized extensive structure and a set of durations. Second, the properties derived under these conditions yield a right action on the generalized extensive structure. Finally, the weighted additive model is obtained as a representation of the generalized extensive structure equipped with the right action.

1. Introduction

Preference for the advanced timing of satisfaction, called *impatience* (Koopmans, 1960), is a well-known concept in the field of intertemporal choice. The concept is illustrated by the following example: receiving \$1000 now is probably preferred to receiving \$1100 after one year. A major reason for this preference is that the value of outcomes decreases with the passage of time.

Koopmans (1972) axiomatized a utility model for infinite outcome sequences to enable it to deal with impatience, which is caused by advancing the receipt of outcomes by "any finite number of periods". By incorporating five postulates consisting of weaker independence, stationarity, [1] and monotonicity (which is different from monotonicity related to a binary operation) into the topological framework of Debreu (1960), the utility model was first constructed on the space of actually finite outcome sequences; then by the use of continuity it was extended to a utility model on the space of infinite outcome sequences. Denoting an infinite-period temporal sequence of outcomes by $(a_1,\ a_2, \ldots)$, his utility model is expressed as the power series $u(a_1,\ a_2, \ldots) = \sum_{i=1}^{\infty} \alpha^{i-1} u(a_i)$, where $0 < \alpha < 1$ is a constant discount factor. Furthermore, Bleichrodt, Rohde, and Wakker (2008) refined Koopmans' formulation theoretically. Aiming to make it possible to deal with unbounded outcome sequences, they weakened the continuity condition from an infinite-dimensional version to a finite-dimensional version by introducing two conditions (constant-equivalence, tail-robustness). This work might make a test of axioms feasible. However, although these theories might be suitable for outcomes expressed by real numbers, e.g., amounts of money, it is too restrictive to treat preferences among qualitative outcomes because the validity of topological conditions (connectedness, separability) is nearly impossible

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1 Stationarity means if the receipt times of two commodities are advanced or deferred by the same amount, the preference between two commodities is invariant.

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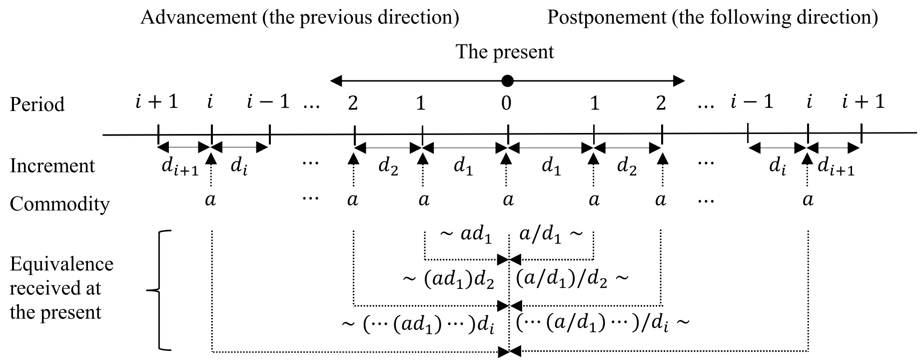

Fig. 1. Advanced receipt and postponed receipt of commodities by right multiplication and right division, respectively: in the case of advanced receipt, $(\cdots(ad_1)\cdots)d_i$ is defined as a solution x to $(x, 0) \sim (\cdots((a, 0)d_1)\cdots)d_i$ (Proposition 5); in the case of postponed receipt, $(\cdots(a/d_1)\cdots)/d_i$ is defined as a solution x to $(x, 0) \sim (\cdots((a, 0)/d_1)\cdots)/d_i$ (Proposition 7).

to test directly. To address the problem, Hübner and Suck (1993) adapted Koopmans' result to a general algebraic framework. That is, they extended the n-component, additive conjoint structure (Krantz, Luce, Suppes, & Tversky, 1971) to an infinite-dimensional version and added stationarity and monotonicity to derive the same utility model as above. Their utility model was similarly constructed using these two steps, apart from the substitution of restricted solvability for continuity in the step of extending to an infinite-dimensional model. They also could deal with unbounded outcome sequences. Meanwhile, Fishburn and Rubinstein (1982) constructed a utility model reflecting impatience in such a different way that it is determined as a function of a single outcome at a particular time. A major advantage of their work is to generalize stationarity to the Thomsen condition. Let X be a set of outcomes and let T be a set of times. Both sets are assumed to be nonnegative real intervals. With the help of the topological framework of Debreu (1960) (Fishburn, 1970), they also derived the multiplicative utility function $u(x, t) = \varphi(t)v(x)$, where v is increasing on X and φ is positive and decreasing on T (due to impatience). However, these two works possess drawbacks. First, as was pointed out by Hübner and Suck (1993), the validity of their conditions is difficult to test because they are formulated in the infinite-dimensional structure. Second, as in Fishburn and Rubinstein's (1982) work, the Thomsen condition is itself artificial, and unfortunately, the problem of topological conditions still remains to be solved.

Recently, Matsushita (2014) constructed a *weighted additive model*, which is an order-preserving function on a generalized extensive structure A that is of the form

$$u(ab) = \alpha u(a) + u(b), \ \alpha \geqslant 1, \tag{1}$$

where ab is the concatenation of a and b, and it implies receiving a one period earlier from now and b now. This is a representation of the generalized extensive structure, called a central left nonnegative concatenation structure with left identity. The left identity element e plays an important role in this construction. The right multiplication (resp. right division) of a by e indicates advancing (resp. postponing) its receipt by one period. Hence $(a/e)b$ implies receiving a and b in the same time period. By defining a new operation as $a \circ b = (a/e)b$, the central left nonnegative concatenation structure reduces to an extensive structure with respect to the operation $\circ$. Using the fact that ab is equivalent to $(ae/e)b$ and letting u be an additive representation of the extensive structure, one obtains $u(ab) = u(ae) + u(b)$; in view of u being a ratio scale, it is possible to derive $u(ae) = \alpha u(a)$. The inequality $\alpha \geqslant 1$ (which is interpreted as a markup factor) shows that the model reflects impatience. Indeed, this construction uses the axioms of r-nonnegativity ($ae \succsim a$) and monotonicity ($a \succsim b \Leftrightarrow ae \succsim be$), which correspond to impatience (in the wider sense) and stationarity, respectively. Since every sequence is expressed as a concatenation, the above model can evaluate preferences between outcome sequences with "any distinct finite number of periods". Hence the model solves the problem involved in Hübner and Suck's (1993) formulation.

However, Matsushita's (2014) formulation has a problem in that the condition of stationarity is used. It is well known (Loewenstein & Prelec, 1992) that stationarity (constant impatience) is often violated. The preference in the first paragraph can be reversed if the delay time is increased with the time lag held constant: receiving \$1100 after three years may be preferred to receiving \$1000 after two years, which is an example of decreasing impatience. Attema, Bleichrodt, Rohde, and Wakker (2010) showed the other type of violation of stationarity – increasing impatience[2] – by analyzing the behavior of subjects faced with intertemporal (delayed) choice problems through time-tradeoff sequences: subjects are increasingly impatient for periods close to the present and constantly impatient for later periods. Moreover, Takahashi, Han, and Nakamura (2012) showed that the exponential discount function with logarithmic time perception, a psychological time duration with the logarithmic unit, is transformed into the generalized hyperbolic discount function (Loewenstein & Prelec, 1992); in other words, perceiving time according to a logarithmic scale and constantly discounting in terms of this perceived time yields decreasing impatience (Attema et al., 2010), because the exponential discount function reflects constant impatience and the hyperbolic one captures decreasing impatience.

These works bring us the following concept: if a one-period advance occurs in a period closer to the present, then a person is sensitive to the advance, and if the person is increasingly (or decreasingly) impatient, then he/she may feel as if its time increment is smaller (or greater) than the actual increment. To allow for the effect of the time duration varying according to a period in which an advance occurs, we express a one-period advance by multiplying outcomes by an increment in a "subjective" duration (not e) corresponding to the period on the right. We then study a utility model reflecting nonconstant impatience under measurement theory. To be more precise, let d_{s_i} be an increment in a subjective duration when advancing the receipt of a from period $i-1$ to period i in the previous direction (see Fig. 1). We express the advanced receipt of a by n periods as $(\cdots(ad_{s_1})\cdots)d_{s_n}$, and construct a utility model of the form

$$u((\cdots(ad_{s_1})\cdots)d_{s_n}) = (\varphi(d_{s_1})\cdots\varphi(d_{s_n}))u(a), \ \varphi(d_{s_i}) > 1,$$

for which u is the weighted additive model of (1), and φ is a weight function of increments in the duration. Since right division

[2] Although theoretical studies commonly assumed decreasing impatience, several empirical studies (Attema, Bleichrodt, Gao, Huang, and Wakker 2016) have found increasing impatience.