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Necessary and possible indifferences

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HIGHLIGHTS

- A necessary and possible indifference is a suitable pair of nested symmetric relations on a set of alternatives.
- The symmetric relations induced by a NaP-preference form a necessary and possible indifference.
- Necessary and possible indifferences are characterized by the existence of a family of equivalence relations.
- Necessary and possible indifference naturally arise in applications, for instance in the field of choice theory.
- We classify necessary and possible indifferences in two types: derived (from a NaP-preference) and primitive.

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ABSTRACT

A NaP-preference (necessary and possible preference) is a pair of nested reflexive relations on a set such that the smaller is transitive, the larger is complete, and the two relations jointly satisfy properties of transitive coherence and mixed completeness. It is known that a NaP-preference is characterized by the existence of a set of total preorders whose intersection and union give its two components. We introduce the symmetric counterpart of a NaP-preference, called a NaP-indifference: this is a pair of nested symmetric relations on a set such the smaller is an equivalence relation, and the larger is a transitively coherent extension of the first. A NaP-indifference can be characterized by the existence of a set of equivalence relations whose intersection and union give its two components. NaP-indifferences naturally arise in applications: for instance, in the field of individual choice theory, suitable pairs of similarity relations revealed by a choice correspondence yield a NaP-indifference. We classify NaP-indifferences in two categories, according to their genesis: (i) derived, which are canonically obtained by taking the symmetric part of a NaP-preference; (ii) primitive, which arise independently of the existence of an underlying NaP-preference. This partition into two classes turns out to be related to the notion of incomparability graph.

1. Introduction

The classical way to represent the (non-stochastic) preference structure of an economic agent on a set of alternatives is by means of a binary relation satisfying suitable order properties, which are usually forms of transitivity and/or completeness. Preorders, semiorders (Luce, 1956; Pirlot & P.Vincke, 1997), and interval orders (Fishburn, 1970, 1985) are the binary relations that are often used for the modelization of preference structures, due to their intrinsic properties: see (Aleskerov, Bouyssou, & Monjardet, 2007; Pirlot & P.Vincke, 1997) and references therein.

A very recent approach to preference modeling employs instead a pair of interconnected binary relations on the same set of alternatives. This bi-preference approach has the advantage of allowing a more flexible modelization of an economic agent's (or a set of economic agents') preference structure in several scenarios. The two preference relations are nested into each other, and are connected by (economically and psychologically) meaningful properties. The main feature of a bi-preference structure is that the two "core properties" of transitivity and completeness are not required to fully hold for both relations, instead they are suitably spread over the combination of the two relations.

NaP-preferences (necessary and possible preferences) belong to the family of bi-preferences. Originally, NaP-preferences were introduced in the field of *Multiple Criteria Decision Aid*, in the process of constructing a new methodology called *Robust Ordinal Regression* (Greco, Mousseau, & Słowiński, 2008). However,

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the axiomatization of this bi-preference structure only came a few years later (Giarlotta & Greco, 2013). Even more recently, NaP-preferences have been studied from several perspectives:

- properties of transitive coherence linking the two components, and their relationship with the genesis of interval orders and semiorders (Giarlotta, 2014);
- more generally, relationships of NaP-preferences with (m, n)-*Ferrers* preference relations – in the sense of Giarlotta and Watson (2014) and Giarlotta and Watson (2017a) – (see also Cantone, Giarlotta, Greco and Watson (2016) for a choice theoretic approach);
- asymmetric and normalized forms of NaP-preferences (Giarlotta, 2015);
- *well-graded* families – in the sense of Doignon and Falmagne (1997) – of NaP-preferences (Giarlotta & Watson, 2017c);
- bi-preference structures under uncertainty (Cerreia-Vioglio, Giarlotta, Greco, Maccheroni & Marinacci, 2017);
- generalizations of NaP-preferences by uniform types of bi-preferences Giarlotta and Watson (2017b);
- necessary and possible *hesitant fuzzy sets* – in the sense of Torra (2010) – to model fuzzy decision problems in a collective setting (Alcantud & Giarlotta, 2017).

Formally, a NaP-preference is a pair of nested reflexive relations on a set of alternatives such that the smaller component is transitive, the larger is complete, and the two components jointly satisfy the properties of transitive coherence and mixed completeness. The smaller component is the *necessary preference*, and codifies the part of a preference structure that is deemed to be at the very core of the "mental attitude" of the economic agent(s): in fact, it collects all relationships among alternatives that *must* happen. The larger component is the *possible preference*, and codifies the "environment" of a preference structure: in fact, it collects all relationships among alternatives that *may* happen. The properties of transitive coherence and mixed completeness are required to hold in order to make the transition between the two components smooth. The naturalness of this bi-preference structure is witnessed by its characterization: a pair of binary relation is a NaP-preference if and only if there exists a family of total preorders, called a *resolution*, whose intersection and union give, respectively, the two components (see Giarlotta & Greco, 2013).

As customary in preference theory, one ought to study both the asymmetric and the symmetric counterparts of the notion of NaP-preference. The analysis of the former has already been carried out in Giarlotta (2015). Here we undertake the study of the symmetric counterpart of a NaP-preference, called a *NaP-indifference*: this is a pair of reflexive and symmetric relations on the same set of alternatives such that the necessary component is transitive, and the possible component is a transitively coherent enlargement of the former. Thus, a NaP-indifference codifies similarity of alternatives in two forms: "strong" (or "mental"), represented by an equivalence relation, and "weak" (or "behavioral"), represented by a suitable symmetric extension of the core relation. Again, the naturalness of this notion is witnessed by its characterization: a pair of symmetric relations is a NaP-indifference if and only if there exists a family of equivalence relations, called a *resolution*, whose intersection and union give the two components (see Theorem 3.4).

We present several examples of NaP-indifferences, and link resolutions of NaP-preferences to those of NaP-indifferences. In particular, we show that natural notions of similarity of items derived from an observed choice behavior – according to Samuelson's classical approach of *revealed preference theory* (Samuelson, 1938) – yield NaP-indifferences on the underlying set of alternatives. Further, we classify NaP-indifferences in two classes, depending on whether they are canonically induced by NaP-preferences or not: we call *derived* the first, and *primitive* the second. The topic of primitive NaP-indifferences turns out to be connected to the primitivity of a single symmetric binary relation, and has therefore a graph-theoretic flavor. Maybe surprisingly, many NaP-indifferences turn out to be primitive, a fact that witnesses the non-redundancy of our analysis.

The paper is organized as follows. In Section 2 we recall the notion of a NaP-preference, and mention some basic results. In Section 3 we introduce the notion of a NaP-indifference, prove a characterization, and give several examples; in particular, we emphasize the links with similarity relations in revealed preference theory. In Section 4 we study the two classes of NaP-indifferences, derived and primitive.

2. Necessary and possible preference structures

In this section, we summarize some – old and new – facts on NaP-preferences. This preliminary overview of the topic is useful to put further developments of the theory of NaP-preferences in the right perspective. The main references for NaP-preferences are (Giarlotta & Greco, 2013) for the seminal paper, and (Alcantud & Giarlotta, 2017; Cerreia-Vioglio et al., 2017; Giarlotta, 2014, 2015; Giarlotta & Watson, 2017c) for further developments.

2.1. Mono-preferences

To start, we recall the basic terminology on binary relations. (The reader may consult (Aleskerov et al., 2007) for relevant definitions and properties.) Unless otherwise specified, hereafter the symbol $\succsim$ denotes a reflexive binary relation on a nonempty set X (of alternatives, courses of actions, etc.). The relation $\succsim$ is generically called a *weak preference*, where $x \succsim y$ stands for "alternative x is at least as good as alternative y". In the special case of an antisymmetric weak preference, we shall often use the symbol $\succcurlyeq$ in place of $\succsim$.

Following standard practice, three binary relations are canonically derived from the primitive weak preference $\succsim$, taking, respectively, its asymmetric part $\succ$, its symmetric part $\sim$, and the symmetric part $\perp$ of its complement. Specifically, we distinguish[1]:

– the *strict preference* $\succ$, defined by $x \succ y$ if $x \succsim y$ and $\neg(y \succsim x)$;
– the *indifference* $\sim$, defined by $x \sim y$ if $x \succsim y$ and $y \succsim x$;
– the *incomparability* $\perp$, defined by $x \perp y$ if $\neg(x \succsim y)$ and $\neg(y \succsim x)$.

Clearly, any weak preference is the (disjoint) union of its strict preference and its indifference. A weak preference $\succsim$ is *complete* (or *total*) if there are no incomparable elements, that is, for each distinct $x, y \in X$, either $x \succsim y$ or $y \succsim x$ (or both) holds.

Typically, a binary relation modeling an economic agent's preference structure is assumed to satisfy the transitivity property, at least partially. The weakest form of transitivity is acyclicity: $\succsim$ is *acyclic* if there is no nontrivial cycle of strict preferences, that is, there are no $n \geq 3$ elements $x_1, x_2, \ldots, x_n \in X$ such that $x_1 \succ x_2 \succ \ldots \succ x_n \succ x_1$. A slightly stronger form of transitivity is quasi-transitivity: $\succsim$ is *quasi-transitive* if its asymmetric part $\succ$ is transitive. Even stronger forms of transitivity (but still weaker than full transitivity) are the Ferrers property and semitransitivity: $\succsim$ is *Ferrers* if $(x \succsim y \wedge z \succsim w)$ implies $(x \succsim w \vee z \succsim y)$, and *semitransitive* if $(x \succsim y \wedge y \succsim z)$ implies $(x \succsim w \vee w \succsim z)$. In fact, both the Ferrers property and semi-transitivity imply quasi-transitivity (and completeness as well), and quasi-transitivity implies acyclicity. Then, a weak preference is called

(1) a *suborder* if it is acyclic,
(2) a *quasi-preorder* if it is quasi-transitive,

[1] Hereafter, variables such as $x, y, z, \ldots$ represent items taken in the set X; as customary, we shall usually omit writing the corresponding universal quantifiers.