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Universal semiorders

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HIGHLIGHTS

- We introduce modified forms of lexicographic products of total preorders, called $\mathbb{Z}$-products.
- We introduce a modified form of the trace of a semiorder, called sliced trace.
- Any semiorder embeds in a $\mathbb{Z}$-product whose extreme factors are the transitive closure and a sliced trace.
- $\mathbb{Z}$-lines, which are $\mathbb{Z}$-products having linear orders as extreme factors, are universal semiorders.
- Rabinovitch's result on the dimension of a strict semiorder is a corollary of our description.

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ABSTRACT

A $\mathbb{Z}$-product is a modified lexicographic product of three total preorders such that the middle factor is the chain of integers equipped with a shift operator. A $\mathbb{Z}$-line is a $\mathbb{Z}$-product having two linear orders as its extreme factors. We show that an arbitrary semiorder embeds into a $\mathbb{Z}$-product having the transitive closure as its first factor, and a sliced trace as its last factor. Sliced traces are modified forms of traces induced by suitable integer-valued maps, and their definition is reminiscent of constructions related to the Scott–Suppes representation of a semiorder. Further, we show that $\mathbb{Z}$-lines are universal semiorders, in the sense that they are semiorders, and each semiorder embeds into a $\mathbb{Z}$-line. As a corollary of this description, we derive the well known fact that the dimension of a strict semiorder is at most three.

1. Introduction

Semiorders are among the most studied categories of binary relations in preference modeling. This is due to the vast range of scenarios which require the modelization of a preference structure to be more flexible and realistic than what a total preorder can provide. On this point, Chapter 2 of the monograph on semiorders by Pirlot and Vincke (1997) gives a large account of possible applications of semiordered structures to various fields of research.

The concept of semiorder originally appeared in 1914 – albeit under a different name – in the work of Fishburn and Monjardet (1992) and Wiener (1914). However, this notion is usually attributed to Luce (1956), who formally defined a semiorder in 1956 as a pair (P, I) of binary relations satisfying suitable properties. The reason that motivated Luce to introduce such a structure was to study choice models in settings where economic agents exhibit preferences with an intransitive indifference. Luce's original definition takes into account the reciprocal behavior of the strict preference P (which is transitive) and the indifference I (which may fail to be transitive). Nowadays, a semiorder is equivalently defined as either a reflexive and complete relation that is Ferrers and semitransitive (sometimes called a *weak semiorder*), or an asymmetric relation that is Ferrers and semitransitive (sometimes called a *strict semiorder*).

Due to the universally acknowledged importance of semiordered structures, several contributions to this field of research have appeared since Luce's seminal work. Many papers on the topic deal with representations of semiorders by means of real-valued functions (Beja & Gilboa, 1992; Campión, Candeal, Induráin, & Zudaire, 2008; Candeal & Induráin, 2010; Candeal, Induráin, & Zudaire, 2002; Gensemer, 1987; Krantz, 1967; Lehrer & Wagner, 1985; Manders, 1981; Monjardet, 1978; Nakamura, 2002), whereas others study the weaker notion of *interval order*,

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introduced by Fishburn (1970), Fishburn (1973) and Fishburn (1985). On the topic of real-valued representations of interval orders and semiorders, a relevant issue is the connection among several notions of *separability*: Cantor, Debreu, Jaffray, strongly, weakly, topological, interval order, semiorder, etc.: on the point, see, e.g., Bosi, Candeal, Induráin, Olóriz, and Zudaire (2001) and Candeal, Estevan, Gutiérrez García, and Induráin (2012) and references therein. The most comprehensive reference on semiorders is the monograph of Pirlot and Vincke (1997). For the relation among utility representations, preferences, and individual choices, we refer the reader to the recent treatise of Aleskerov, Bouyssou, and Monjardet (2007).[1] Semiordered structures have also been studied in the context of the assessment of knowledge and the construction of a teaching engine: concerning these topics, see the monographs on *knowledge spaces* (Doignon & Falmagne, 1999) and *learning spaces* (Falmagne & Doignon, 2011), as well as the notion of a well-graded family of binary relations (Doignon & Falmagne, 1997).

In 1958 Scott and Suppes (1958) tried to identify a semiorder by means of the existence of a *shifted* real-valued utility function u, in the following sense: xPy (to be read as "alternative y is strictly preferred to alternative x") holds if and only if $u(x) + 1 < u(y)$. In this representation, the real number 1 is to be intended as a "threshold of perception or discrimination", which gives rise to the so-called *just noticeable difference* (Manders, 1981). The shifted utility function u is classically referred to as a *Scott–Suppes representation* of the semiorder.

It is well known that not every semiorder admits a Scott–Suppes representation. In fact, as Ŝwistak points out in Ŝwistak (1980), the existence of a Scott–Suppes utility function imposes strong restrictions of the structure of a semiorder. However, this type of representation has been given a lot of attention over time, due to its importance in several fields of research, such as extensive measurement in mathematical psychology (Krantz, 1967; Lehrer & Wagner, 1985), choice theory under risk (Fishburn, 1968), decision-making under risk (Rubinstein, 1988), modelization of choice with errors (Agaev & Aleskerov, 1993), etc.[2]

Scott and Suppes (1958) showed that every finite semiorder always admits such a representation (see also Rabinovitch, 1977). In 1981 Manders (1981) proved that – under a suitable condition related to the non-existence of monotone sequences with an upper bound in the set (a property later on called *regularity*) – countable semiorders have a Scott–Suppes representation as well. A similar result was obtained in 1992 by Beja and Gilboa (1992), who introduced new types of representations – *GNR* and *GUR*, having an appealing geometric flavor – of both interval orders and semiorders.

Following a stream of research providing "external" characterizations of Scott–Suppes representable semiorders (Candeal et al., 2002), in 2010 Candeal and Induráin (Candeal & Induráin, 2010) obtained what they call an "internal" characterization of the Scott–Suppes representability of an arbitrary semiorder. Their characterization uses both regularity and *s-separability*, the latter being a condition similar to the Debreu-separability of a total preorder but involving the trace of the semiorder.[3]

There are many additional studies on semiorders, most of which however restrict their attention to the finite case. As a matter of fact, the monograph on semiorders (Pirlot & Vincke, 1997) is almost entirely dedicated to finite semiorders, due to the intrinsic difficulties connected to the analysis of the infinite case.[4] Among the studies that concern infinite semiorders, let us mention the work of Rabinovitch (1978), who proved in 1978 that the *dimension* of a strict semiorder is at most three (that is, the asymmetric part of a semiorder can be always written as the intersection of three strict linear orders).

In this paper, we describe the structure of an arbitrary semiorder, regardless of its size. In fact, we obtain a universal type of semiorder, in which every semiorder embeds (Theorem 5.6). These universal semiorders are suitably modified forms of lexicographic products of three total preorders. The modification is determined by a shift operator, which typically creates intransitive indifferences. Since the middle factor of these products is always the standard linear ordering $(\mathbb{Z}, \leq)$, and the shift operator is applied to it, we call these modified lexicographic structures $\mathbb{Z}$*-products*. In particular, we prove that $\mathbb{Z}$*-lines*, which are the $\mathbb{Z}$-products having linear orders as their extreme factors, are universal semiorders as well (Corollary 5.7).

Our results on semiorders are related to a general stream of research that uses lexicographic products to represent preference relations. In this direction, the literature in mathematical economics has been mainly focused on lexicographic representations of well-structured preferences, which assume the form of total preorders or linear orders. Historically – following some order-theoretic results of Hausdorff (1914) and Sierpiński (1940) concerning representations by means of lexicographically ordered transfinite sequences – Chipman (1971) and Thrall (1954) were the first authors to develop a theory of lexicographic preferences. Among the several important contributions that followed, let us recall the structural result of Beardon, Candeal, Herden, Induráin, and Mehta (2002), which provides a subordering classification of all chains that are *non-representable* in $\mathbb{R}$ (that is, they cannot be order-embedded into the reals).[5] The (dated but always valuable) survey of Fishburn (1974) provides a good source of references on lexicographic representations of preferences.[6]

The results on lexicographic structures mentioned in the previous paragraph describe linear orders in terms of universal linear orders. The main result of this paper has a similar flavor, since it describes semiorders in terms of universal semiorders, that is, $\mathbb{Z}$-products (and, in particular, $\mathbb{Z}$-lines). In the process of obtaining such a representation, we explicitly construct a special $\mathbb{Z}$-product in which a given semiorder embeds (Theorem 5.6(iv)). The procedure that allows us to differentiate the elements of a semiordered structure can be summarized as follows:

1 On individual choice theory and the associated theory of revealed preferences, see also (Cantone, Giarlotta, Greco, & Watson, 2016) (and references therein), where the authors develop an axiomatic approach based on the satisfaction of the so-called *weak*(m, n)*-Ferrers properties*, recently introduced by Giarlotta and Watson (2014b) (which include semiorders as particular cases, that is, binary relations that are both weakly (2, 2)-Ferrers and weakly (3, 1)-Ferrers).

2 See Abrísqueta et al. (2012) for a very recent survey on the Scott–Suppes representability of a semiorder.

3 By *external* the authors mean that the characterization is based on the construction of suitable ordered structures that are related to the given semiorder. On the other hand, *internal* means that the characterization is entirely expressed in terms of structural features of the semiorder.

4 To further emphasize this point, note that the First Edition (2002) of the treatise of Aleskerov et al. (2007) on utility maximization, choice and preference was almost entirely dedicated to covering the analysis of the finite case. This is the main reason why a Second Edition of the book appeared in 2007. In fact, Chapter 6 of Aleskerov et al. (2007) is now entirely dedicated to preference representation theory for the infinite case (in particular, infinite semiorders).

5 The mentioned result directly involves a basic prototype of lexicographic product, namely, the lexicographically ordered real plane $\mathbb{R}^2_{\text{lex}}$. (Note that $\mathbb{R}^2_{\text{lex}}$ is the example used by Debreu (1954) in his famous paper on the *Open Gap Lemma* to disprove the inveterate belief that ordered preferences admit a real-valued utility representation.) Beardon et al. (2002) prove the following: A linear ordering is non-representable in $\mathbb{R}$ if and only if it is either (i) *long* (i.e., it contains a copy of the first uncountable ordinal ω_1 or its reverse ordering ω_1^*), or (ii) *large* (i.e., it contains a copy of a non-representable subset of $\mathbb{R}^2_{\text{lex}}$), or (iii) *wild* (i.e., it contains a copy of an *Aronszajn line*, which is an uncountable chain such that neither ω_1 nor ω_1^* nor an uncountable subchain of the reals embeds into it.)

6 For recent contributions on the topic, the reader may consult (Candeal & Induráin, 1999; Giarlotta, 2004, 2005; Giarlotta & Watson, 2009, 2013, 2014a; Knoblauch, 2000) and references therein.