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Bi-semiorders with frontiers on finite sets[☆]

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HIGHLIGHTS

- We extend the notion of bi-semiorders to cope with frontiers.
- We study the numerical representation of bi-semiorders with frontiers.
- We relate the problem to conjoint measurement.

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ABSTRACT

This paper studies an extension of bi-semiorders in which a "frontier" is added between the various relations used. This extension is motivated by the study of additive representations of ordered partitions and coverings defined on product sets of two components.

1. Introduction

Let $\mathcal{T}$ be a relation between two sets A and Z, i.e., a subset of $A \times Z$. *Biorders* are relations between two sets that lead to a numerical representation in which there are real-valued functions f on A and g on Z such that, for all $a \in A$ and all $p \in Z$,

$$a \mathcal{T} p \Leftrightarrow f(a) > g(p).$$

The name "biorder" comes from Doignon, Ducamp, and Falmagne (1984) and has gained wide acceptance (see Doignon, Ducamp, & Falmagne, 1987; Nakamura, 2002). This structure was introduced in the literature by Riguet (1951) who used the term "Ferrers relation". It was studied by Ducamp and Falmagne (1969) under the name "bi-quasi-series". Early work on biorders include Bouchet (1971) and Cogis (1976, 1982a,b) (see Monjardet, 1978; Doignon & Falmagne, 1999, p. 60, for a detailed historical account).

Biorders are useful to model Guttman scales (Guttman, 1944, 1950). They are also an important tool to study various classes of binary relations, most notably *interval orders* and *semiorders* (Aleskerov, Bouyssou, & Monjardet, 2007; Fishburn, 1985; Pirlot & Vincke, 1992). Indeed, when $A = Z$, an irreflexive biorder is nothing but an interval order, as defined in Fishburn (1970). Adding semitransitivity to irreflexivity leads to semiorders (Luce, 1956; Scott & Suppes, 1958).

In Bouyssou and Marchant (2011) (henceforth, BM11), we have studied an extension of biorders in which there are *two* relations $\mathcal{T}$ and $\mathcal{F}$ between the sets A and Z, leading to what we called *biorders with frontier*. They lead to a numerical representation in which there are real-valued functions f on A and g on Z such that, for all $a \in A$ and all $p \in Z$,

$$a \mathcal{T} p \Leftrightarrow f(a) > g(p),$$
$$a \mathcal{F} p \Leftrightarrow f(a) = g(p).$$

With *bi-semiorders*, we have two relations $\mathcal{T}$ and $\mathcal{P}$ between the sets A and Z. The numerical representation involves a real-valued function f on A and a real-valued function g on Z such that, for all

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$a \in A$ and $p \in Z$,

$a\mathcal{P}p \Leftrightarrow f(a) > g(p) + 1,$

$a\mathcal{T}p \Leftrightarrow f(a) > g(p).$

Necessary and sufficient conditions for the above model were given in Ducamp and Falmagne (1969, Th. 5) when both A and Z are finite sets (note that the term bi-semiorder is used in Fishburn, 1997, with a different meaning).[1]

Bi-semiorders with frontiers will use four relations $\mathcal{P}$, $\mathcal{J}$, $\mathcal{T}$ and $\mathcal{F}$ between the sets A and Z. The numerical representation involves a real-valued function f on A and a real-valued function g on Z such that, for all $a \in A$ and $p \in Z$,

$a\,\mathcal{P}\,p \Leftrightarrow f(a) > g(p) + 1,$

$a\,\mathcal{J}\,p \Leftrightarrow f(a) = g(p) + 1,$

$a\,\mathcal{T}\,p \Leftrightarrow f(a) > g(p),$

$a\,\mathcal{F}\,p \Leftrightarrow f(a) = g(p).$

The purpose of this paper is to establish necessary and sufficient conditions for the above model when both A and Z are finite sets.

The paper is organized as follows. Section 2 briefly presents our initial motivation for studying structures with frontiers. Section 3 presents our setting. Results on biorders, biorders with frontier and bi-semiorders are recalled in Section 4. Section 5 presents our results on bi-semiorders with frontiers that are proved in Section 6.

2. Relation to conjoint measurement

2.1. Additive representations of ordered coverings

Our initial motivation for studying biorders and bi-semiorders with frontiers is linked to the following problem. Let $X = X_1 \times X_2 \times \cdots \times X_n$ be a set of objects evaluated on n attributes.

Suppose that we are given an ordered *covering* $\langle C^1, C^2, \ldots, C^r\rangle$ of the set of objects. In such a setting, we know that objects belonging to C^{k+1} are better than objects belonging to C^k but we have no information on the way two objects belonging to the same category compare in terms of preference. The category C^k can have a nonempty intersection with C^{k+1} and C^{k-1}. Its intersection with other categories is always empty, reflecting the ordered nature of the covering.

Consider first an ordered *partition* $\langle C^1, C^2, \ldots, C^r\rangle$. In this case, we are interested in finding real-valued functions u_i on X_i such that, for all $x \in X$ and all $k \in \{1, 2, \ldots, r\}$,

$$x \in C^k \Leftrightarrow \sigma^{k-1} < \sum_{i=1}^{n} u_i(x_i) \leq \sigma^k, \tag{1}$$

with the convention that $\sigma^0 = -\infty$, $\sigma^r = +\infty$ and where $\sigma^1, \sigma^2, \ldots, \sigma^{r-1}$ are real numbers such that $\sigma^1 < \sigma^2 < \cdots < \sigma^{r-1}$. In the case of an ordered *covering* $\langle C^1, C^2, \ldots, C^r\rangle$, the model becomes

$$x \in C^k \Leftrightarrow \sigma^{k-1} \leq \sum_{i=1}^{n} u_i(x_i) \leq \sigma^k, \tag{2}$$

so that, if $\sum_{i=1}^{n} u_i(x_i) = \sigma^{k-1}$, the object x belongs at the same time to C^{k-1} and to C^k, i.e., is at the frontier between these two categories.

[1] The fact that two thresholds, the first one at 1 and the other one at 0, are used in the numerical representation of bi-semiorders may lead one to think that there is a link with the study of families of semiorders having a constant threshold representation (see Cozzens & Roberts, 1982, Roubens & Vincke, 1985, ch. 6), Roy & Vincke, 1987 for the case of a family of two semiorders and Doignon, 1987, for the general case). This is misleading. Indeed, Ducamp and Falmagne (1969) have shown that a bi-semiorder is the natural counterpart of a structure involving a *single* semiorder when studying relations between two different sets.

The analysis of the above models in the general case requires the use of conjoint measurement techniques (see Bouyssou & Marchant, 2009, 2010, following initial results by Fishburn, Lagarias, Reeds, & Shepp, 1991 and Vind, 1991, 2003).

However, as suggested by the results of Levine (1970), there are some particular cases that can be dealt with in a simpler way. Biorders are useful to study the case of a product set with *two* components and an ordered *partition* with *two* categories. Biorders with frontiers are useful to deal with the case of a product set with *two* components and an *ordered covering* with *two* categories. We mentioned in BM11 Sect.7, that the case of *three* ordered categories and a product set with *two* components was also quite particular. When the three ordered categories partition the product set, we can indeed use the results on *bi-semiorders* presented in Ducamp and Falmagne (1969, Sect. IV) (see also Ducamp, 1978). The results presented in this paper allows us to deal with the case in which the three ordered categories are a covering, instead of a partition, of the product set.

2.2. Particular cases with two attributes

Consider first the case of ordered partitions of $X = X_1 \times X_2$.

When there are only two attributes and two categories, the additive representation (1) relates more to ordinal than to conjoint measurement. Indeed, in such a case, the problem clearly reduces to finding real-valued functions u_1 on X_1 and u_2 on X_2 such that, for all $x = (x_1, x_2) \in X$,

$$x \in C^2 \Leftrightarrow u_1(x_1) + u_2(x_2) > \sigma. \tag{3}$$

It is easy to see that it is not restrictive to suppose that $\sigma = 0$. Define the relation $\mathcal{T}$ between the sets X_1 and X_2 letting, for all $x_1 \in X_1$ and all $x_2 \in X_2$,

$x_1\mathcal{T}x_2 \Leftrightarrow (x_1, x_2) \in C^2.$

It is clear that asking for a representation in model (3) is equivalent to asking for the existence of two functions f on X_1 and g on X_2 such that

$x_1\mathcal{T}x_2 \Leftrightarrow f(x_1) > g(x_2).$

This explains the link with *biorders*.

Similarly, when there are only two attributes and three categories, building an additive representation (1) reduces to finding real-valued functions u_1 on X_1 and u_2 on X_2 such that, for all $x \in X$,

$$\begin{aligned} x \in C^3 \quad &\Leftrightarrow \lambda < u_1(x_1) + u_2(x_2), \\ x \in C^2 \cup C^3 &\Leftrightarrow \rho < u_1(x_1) + u_2(x_2), \end{aligned} \tag{4}$$

where ρ, λ are two thresholds such that $\rho < \lambda$. As detailed in Ducamp and Falmagne (1969), it is not restrictive to suppose that $\rho = 0$ and $\lambda = 1$.

Define the relations $\mathcal{P}$ and $\mathcal{T}$ between the sets X_1 and X_2 letting, for all $x_1 \in X_1$ and all $x_2 \in X_2$,

$x_1\mathcal{P}x_2 \Leftrightarrow (x_1, x_2) \in C^3.$

$x_1\mathcal{T}x_2 \Leftrightarrow (x_1, x_2) \in C^2 \cup C^3.$

It is clear that asking for a representation in model (4) is equivalent to asking for the existence of two functions f on X_1 and g on X_2 such that

$x_1\mathcal{P}x_2 \Leftrightarrow f(x_1) > g(x_2) + 1,$

$x_1\mathcal{T}x_2 \Leftrightarrow f(x_1) > g(x_2).$

This explains the links with *bi-semiorders*.

We now turn to the case of ordered coverings of $X = X_1 \times X_2$.

Suppose first that there are only two categories C^2 and C^1. Allowing for an hesitation between C^2 and C^1 leads to a model in