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Generalization of extensive structures and its representation

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HIGHLIGHTS

- A generalization of extensive structures and its representation are considered.
- A left nonnegative concatenation structure with left identity is defined.
- This structure satisfies solvability and Archimedeaness with left-concatenation.
- Two conditions make the structure into an extensive structure with identity.
- We get the weighted additive model as a representation on the extensive structure.

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ABSTRACT

This paper generalizes extensive structures so that a weighted additive model can be obtained. A left nonnegative concatenation structure with left identity is defined as a nonnegative concatenation structure (Luce et al., 1990) with left identity for which the solvability and Archimedean properties are satisfied only related to left-concatenation. This structure has two partial binary operations – multiplication and right division – and a new partial binary operation is defined on it. Two conditions of equivalence form are then provided to make the left nonnegative concatenation structure with left identity into an extensive structure with identity with respect to the newly defined operation. Finally, the weighted additive model is derived from an additive representation on the extensive structure, so that distinct m-period and n-period ($m \neq n$) temporal sequences can be compared.

1. Introduction

Matsushita (2011) recently generalized the classical result of Hölder (1901) in the context of groupoids (a "groupoid" is a nonempty set with a binary operation), and developed an axiom system to construct a weighted additive model. From groupoid multiplication, let ab denote the concatenation of commodities a, b. Then his model is of the following form:

$$u(ab) = \alpha u(a) + u(b), \quad \alpha \geqslant 1.$$

The first aim of this paper is to convert his algebraic axioms into a decision-making version so that they can be empirically tested. Meanwhile, all axioms, including the remaining ones, are to be rewritten under the requirement that the multiplication be generalized to a partial binary operation, that is, a generalization of extensive structures. Although the framework for constructing the weighted additive model is almost identical to the proof of Theorem 4.2 (Matsushita, 2011), the addition of some mathematical work is needed to achieve this aim. First, two axioms A8 and A9 (Lemma 1), written in a simple form, are proposed from which one can deduce the algebraic axioms. Second, the concepts of extensive "substructure" and "order-isomorphism" (Lemma 3) are introduced to yield the multiplicative form $\alpha u(a)$ in the weighted additive model.

We shall now consider preferences over temporal sequences of amounts of money. Many people will probably prefer receiving \$10,000 this year and \$5000 next year to receiving \$5000 this year and \$10,000 next year. A major reason for this preference is that the value of commodities decreases with the passage of time. Utility models has been already proposed to explain this kind of preference. The simplest one is of the following form: letting $(a_1, \ldots, a_n)$ denote an n-period temporal sequence,

$$\phi(a_1, \ldots, a_n) = \sum_{i=1}^{n} \lambda^{i-1} v(a_i),$$

where ϕ and v are real-valued functions on the set of temporal sequences consisting of n commodities and on the set of single commodities, respectively, and $\lambda \leqslant 1$ is a discount factor at a constant rate.

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The axiomatization to construct this utility model was initiated by Krantz, Luce, Suppes, and Tversky (1971) and Fishburn (1970). They developed a utility model with various discount factors so as to reflect the concept for a preference for advancing the timing of future satisfaction (i.e., impatience; Koopmans, 1960; Koopmans, Diamond, & Williamson, 1964). Then incorporating "stationarity[1]" by Koopmans (1960), they reduced the utility model to the above special model with a discount factor at a constant rate. For this construction, Krantz et al. assumed an "additive conjoint structure" and Fishburn considered a finite product of topological spaces. As such, the following problem arose: comparisons could be made only between temporal sequences with the same number of periods. Further, some of their axioms are difficult to empirically test. Indeed, the n-factor independence condition requires us to consider the ordering of the joint effect of multiple factors in verifying its validity; the validity of the topological conditions (connectedness, separability) is, in itself, nearly impossible to directly test, because it is difficult to have subjects recognize the concept of open or closed sets in the frame of a preference structure.

Our weighted additive model (displayed in the first paragraph) too can deal with multi-period temporal sequences. Identifying $(a_1, \ldots, a_n)$ with $(\cdots((a_1a_2)a_3)\cdots a_{n-1})a_n$, from the inductive use of the equation of the weighted additive model, we have $u[(\cdots((a_1a_2)a_3)\cdots a_{n-1})a_n] = \sum_{i=1}^{n} \alpha^{n-i}u(a_i)$. It should be noted that this is a representation for multiplication. Since every temporal sequence (consisting of any number of commodities) is expressed as a product, this model can numerically evaluate preferences between distinct m-period and n-period ($m \neq n$) temporal sequences. This is a great advantage of our model over the above utility model with a stationary discount factor. Furthermore, in connection with the first aim of the paper, the axioms are to be written as equivalences between commodities or concatenations so that their validity can be empirically tested. Thus, the axiomatization of our weighted additive model offers a solution to the problems raised above. Another marked difference between these two models is that the weight of our model is $\alpha \geqslant 1$, which may be referred to as a markup factor at a constant rate. However, the concepts of a discount factor and a markup factor could be deemed relative, because for one temporal sequence, the receipt of each component is considered postponed or advanced depending on whether one is regarding the oldest or the latest period as a standard; and whether a utility model has a discount factor or a markup factor is determined on the basis of whether one counts each period number in the temporal sequence toward the future direction or toward the past direction. As such, our utility model can explain a preference property, such as impatience. From the above, the second aim of the paper is to put an interpretation on several axioms in the context of the decision-making problems of temporal sequences.

The rest of this paper is organized as follows. Section 2 provides the axioms to define a basic structure, called left nonnegative concatenation structure with left identity, the positive part of which is a generalized concept of a PCS (Luce, Krantz, Suppes, & Tversky, 1990) in the sense that the solvability and Archimedean properties are satisfied only related to left-concatenation. Moreover, some properties are shown to be satisfied on the structure. Section 3 presents two axioms of equivalence form to make every left nonnegative concatenation structure with left identity an extensive structure with identity related to an introduced operation, interprets the axioms in the context of temporal sequences, and gives the main theorem for the weighted additive model. Section 4 contains several conclusions. The proofs of the lemmas, propositions, and theorem are given in Section 5.

[1] Stationarity means that preferences are invariant over temporal sequences $(a_1, \ldots, a_n)$ under the shifts in which each component a_i is advanced or postponed by one period.

2. Basic concepts

Throughout this paper, $\mathbb{R}_0^+$ denotes the set of all nonnegative real numbers. Let $\succsim$ be a binary relation on a nonempty set A that is interpreted as a preference relation. As usual, $\succ$ denotes the asymmetric part, $\sim$ the symmetric part, and $\precsim$, $\prec$ denote reversed relations. The binary relation $\succsim$ on A is a *weak order* if and only if it is connected and transitive. Let $\cdot$ be a "partial" binary operation on A. The operation means a function from a subset B of $A \times A$ into A. The expression $a \cdot b$ is said to be *defined* (in A) if and only if $(a, b) \in B$. An element $e \in A$ denotes no change in the status quo with temporal sequences. That is, it is assumed that receiving e prior to a is no different from receiving a at present; however, ae implies advancing the receipt of a by one period, so that ae is not always $\sim a$.

In the following conditions, all the products are always assumed to be defined.

A1. Weak order: $\succsim$ is a weak order on A.
A2. Local definability: if $a \cdot b$ is defined, $a \succsim c$, and $b \succsim d$, then $c \cdot d$ is defined.
A3. Monotonicity: $a \succsim b \Leftrightarrow a \cdot x \succsim b \cdot x \Leftrightarrow x \cdot a \succsim x \cdot b$ for all $a, b, x \in A$.
A4. Left identity: e is a *left identity element*; that is, $e \cdot a \sim a$ for all $a \in A$.

The system $\langle A, \succsim, \cdot \rangle$ is a *concatenation structure* if and only if A1–A3 are satisfied. If, in addition, A4 holds, then $\langle A, \succsim, \cdot, e \rangle$ is said to be a *concatenation structure with left identity*. Throughout the paper, the trivial case where A has just a single element e is always excluded.

We now state a terminology important to this paper. An element a of a concatenation structure A is *r-nonnegative*, *l-nonnegative*, or *nonnegative* according as $x \cdot a \succsim x, a \cdot x \succsim x$, or both hold for all (x, a) or $(a, x) \in B$. Similarly, *r-positive*, *l-positive*, and *positive* elements can be defined by replacing $\succsim$ with the strict preference relation $\succ$. A concatenation structure is called r-nonnegative if all of its elements are r-nonnegative, and so on.

Fundamental conditions for concatenation structures are listed below.

A5. R-nonnegativity: whenever $x \cdot a$ is defined, then $x \cdot a \succsim x$.
A6. Left solvability: whenever $a \succ b$, there exists $x \in A$ such that $x \cdot b$ is defined and $a \sim x \cdot b$.

Axiom A5 is defined as the "right sided" concept, whereas A6 is defined as the "left sided" concept. That is, r-nonnegativity is the nonnegativity condition that is satisfied only for right-concatenation by a. Left solvability is a generalized solvability in the sense that only the existence of a left solution is permissible. If a concatenation structure contains a left identity element e, then by A3, a is l-positive (or l-nonnegative) if and only if $a \succ e$ (or $a \succsim e$), whereas $a \succ e$ is not always r-positive nor even r-nonnegative (see Example 1). However, the following holds.

Proposition 1. *Let $\langle A, \succsim, \cdot, e \rangle$ be a concatenation structure with left identity. If A is r-nonnegative, then $a \succsim e$ for all $a \in A$.*

Since, in A6, x is uniquely determined up to $\sim$ by A3, we write $x \sim a/b$, and $a/a \sim e$ because $a \sim e \cdot a$. Thus a partial binary operation $/$ is defined on A, which is called a *right division*. Indeed, $/$ is a function from the subset $\{(a, b) \in A \times A \,|\, a \succsim b,\ (x, b) \in B$ for some $x \in A\}$ into A. It may be suitable to refer to A6 as *right divisibility*.

Proposition 2. *Let $\langle A, \succsim, \cdot, e \rangle$ be a concatenation structure with left identity. If* A6 *holds, then for all $a, b, x \in A$, the following properties hold:*

(i) $(a \cdot b)/b \sim a \sim (a/b) \cdot b$ *whenever $a \cdot b$ is defined and $a \succsim b$.*