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What do we know about reasoning and proving? Opportunities and missing opportunities from curriculum analyses☆

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A R T I C L E I N F O

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By examining findings and research methodology across studies focused on reasoning-and-proving in mathematics textbooks, this paper provides commentary on the nature of reasoning and proving and curriculum analysis in mathematics education. Large variations across the studies were noted with regard to the framings of reasoning-and-proving as well as the analytic methods used by each authoring team. Working toward achieving greater methodological consistency is a challenging endeavor that could ultimately inform the field about the quantity and quality of students' opportunities to learn reasoning and proving across the intended mathematics curriculum. We offer suggestions for future research related to better defining and conceptualizing the notions of "reasoning" and "proving", as well as analyzing student learning opportunities beyond the intended curriculum analysis.

Two high school teachers who never met, and who lived in different parts of the United States, were both observed using swimming pool metaphors when introducing proof in their geometry classes. Matt, who claimed that he did not know of any strategies for teaching proof beyond simply *showing* proofs to his students (see Cirillo, 2011, for more information), said the following after completing a few short proofs in his class:

> Okay, there's only so many of these I can do with us together. I just kind of have to keep throwing you in the deep end, letting you thrash around for a while and then throw you a floaty, haul you back out and then throw you back in, alright?

Similarly, Mike, a seasoned geometry teacher, told his students:

> Here we go. So proofs are tough. You know one thing about proofs is there's no easy way. There's no way to do it, like, there's no shallow end. You can't wade into the proof pool. You gotta kind of jump right in the deep end with these tough ones.

It was interesting to observe that even in the geometry course, the place in the curriculum where formal reasoning (i.e., proof) has historically been taught (Clements, 2003), both Matt and Mike were unsure about the best way to introduce proof

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to their students. Both teachers, who had previously taught geometry and were considered well-prepared in terms of their mathematical content knowledge, claimed that they did not have strategies for scaffolding the introduction to proof. It is likely even more interesting to readers of this Special Issue that Matt and Mike both pointed to their textbooks as hampering their introduction to proof. For example, Matt objected to the "presentation style" of his conventional textbook which displayed theorems as finished results in text boxes (Cirillo, Drake, & Herbel-Eisenmann, 2009). He ultimately supplemented and adapted the written curriculum to meet his goal of teaching "real math" that "makes sense". Similarly, after participating in a professional development program on teaching proof in geometry, Mike and his colleague decided not to issue the conventional textbooks provided by the school to their students. Instead, they wrote their own curriculum which they now use to teach proof in their geometry course (Cirillo, 2013).

The empirical data provided above point to an important motivation for conducting analyses like those compiled in this Special Issue on reasoning-and-proving in the written curriculum. In particular, if the field is to heed the call to increase and improve the attention given to reasoning and proving in school mathematics (see NCTM, 1989, 2000), it is important to provide teachers with tools that support this goal. Because features of curriculum materials may act as affordances or constraints in different situations and for different teachers (Lloyd, 2008), curriculum analyses can be an important first step toward improving the opportunities that students have to learn reasoning and proving.

In this commentary, we begin with a discussion of curriculum analysis as a line of scholarly inquiry to situate our discussion of reasoning and proving in textbooks. In the second section, we then discuss definitions and conceptualizations of "reasoning" and "proving" as two important mathematical activities in general, and reasoning-and-proving as a research construct in particular. We sometimes use the hyphenated term, reasoning-and-proving, to reference the research construct corresponding to authors' definitions in this Special Issue. At the same time, throughout this commentary we use the words "reasoning" and "proving" without the hyphenation, referring to two separate, but related activities described by NCTM (2000). In particular, when we refer to *proving*, we reference the process of constructing a proof, where proof is a formal way of expressing particular kinds of reasoning and justification. More specifically, proving is the process of developing arguments consisting of logically rigorous deductions of conclusions from hypotheses (NCTM, 2000). While proving may be a part of reasoning, *reasoning* generally refers to the process of developing arguments, which is not necessarily rigorous deduction.

In the third section, we demonstrate the insightful information curriculum analysis can provide for curriculum developers, policy makers, mathematics specialists, and classroom teachers. In the fourth section, we discuss methodological considerations in curriculum analysis, in general, and reasoning-and-proving in various curricula from the five studies, in particular. We conclude this commentary by offering suggestions for future research related to better defining and conceptualizing the notions of "reasoning" and "proving", as well as analyzing student learning opportunities beyond intended curriculum analyses.

1. Curriculum analysis as a line of inquiry

While there is a complex interplay among teachers, students, and curriculum, there is consensus that curriculum matters for student learning (Cai, Wang, Moyer, Wang, & Nie, 2011b; Schmidt et al., 2002; Stein, Remillard, & Smith, 2007). It matters because it influences what students learn, when they learn it, and how well they learn it. In fact, around the globe, changing the curriculum has been viewed and used as an effective way to change classroom practice and to influence student learning to meet the needs of an ever-changing world (Cai & Howson, 2013; Howson, Keitel, & Kilpatrick, 1981; Senk & Thompson, 2003). Thus, it is not surprising that curriculum has been called a "change agent" for educational reform (Ball & Cohen, 1996). Today, the school mathematics curriculum remains a central issue in efforts to improve student learning. Because of the important role of curriculum in student learning, curriculum analysis has long been a line of scholarly inquiry (Cai, 2010).

In curriculum analysis, "curriculum" usually refers to the intended curriculum, which specifies goals, topics, sequences, instructional activities, and assessment methods and instruments. The most common method of evaluating an intended curriculum is content analysis, which involves judging the quality of the content of a curriculum and the quality of its presentation (NRC, 2004).

The largest scale and most systematic curriculum analyses trace back to the First International Mathematics Study (FIMS) by the International Association for the Evaluation of Educational Achievement (IEA), followed by the Second International Mathematics Study (SIMS) (Husen, 1967; Robitaille & Garden, 1989; Travers & Westbury, 1989). These international studies examined the mathematical topics covered in curricula from different countries. Since then, many researchers have focused on various aspects of curriculum, such as grade level placement of topics, in order to understand the kinds of learning opportunities created for students. For example, Fuson, Stigler, and Bartsch (1988) reported that addition and subtraction were introduced relatively late in U.S. mathematics textbooks compared to textbooks from the former Soviet Union, Taiwan, Mainland China, and Japan. However, there was remarkable uniformity in the placement of addition in the curricula of those four countries.

Some earlier studies also analyzed the sequence of mathematical topics in curricula (e.g., Resnick et al., 1989). Recent trends in curriculum analysis involve fine-grained analyses on how certain mathematical concepts are introduced as well as fine-grained analyses of the mathematical problems included in curricular materials (Cai, Lo, & Watanabe, 2002; Cai, Nie, & Moyer, 2010; Fan & Zhu, 2007; Li, 2000; Nie, Cai, & Moyer, 2009). Problems govern not only students' attention to particular aspects of content, but also their ways of processing information. Thus, analyses of problems included in curriculum materials provide information about one aspect of the kinds of learning opportunities students have (e.g., Doyle, 1988).