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On the construction of set-based meanings for the truth of mathematical conditionals

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A R T I C L E I N F O

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A B S T R A C T

This paper presents a case study of Hugo's construction of Euler diagrams to develop set-based meanings for the truth of mathematical conditionals. We use this case to set forth a framework of three stages of activity in students' guided reinvention of mathematical logic: reading activity, connecting activity, and fluent activity. The framework also categorizes various forms of connecting activity by which students may reflect on their reading activity: connecting tasks, connecting representations, and connecting conditions for truth and falsehood (which we call meanings). We argue that coordinating such connections is necessary to justify logical equivalences, such as why contrapositive statements share truth-values. Through the case study, we document the representations and meanings that Hugo called upon to assign truth-values to conditionals. The framework should help clarify and advance future research on the teaching and learning of logic rooted in students' mathematical activity.

1. Introduction

Though mathematicians such as Frege, Russell, and Tarski developed modern formalizations of mathematical logic less than 150 years ago, their accounts of how mathematical language works have become integral to the way the mathematical community understands the language of proof-oriented mathematics. Because students must abide by these formal conventions of language to some degree, many transition to proof courses teach mathematical logic. However, the mathematics education community's accounts of how students should learn logic for mathematics are still emerging (e.g. Antonini, 2001; Durand-Guerrier, 1996, 2003; Hawthorne & Rasmussen, 2015; Yopp, 2017).

The afore-mentioned mathematicians had the benefit of reflecting on their rich experience reading and writing proofs when constructing logic. We cannot assume the same background for mathematics students whose experience with proving often is minimal and formulaic. In this paper, we set forth a framework of three stages through which students may construct conscious meanings for particular aspects of mathematical language such as the truth conditions of conditionals and the logical equivalence between contrapositive conditionals. The framework begins where novice students begin: with *reading* mathematical statements – *reading activity*. Next, the largest portion of the paper describes a range of processes in the second stage – *connecting activity*. In this stage, students begin reflecting on their reading activity and making their meanings for truth and falsehood explicit. We exemplify the processes of connecting activity using a strategically chosen case study of Hugo's learning about mathematical conditionals and contrapositive equivalence. Third, we point to the presence of a third stage in which the various meanings become interconnected and systematized to support the *fluent* activity with mathematical language exhibited by mathematicians – *fluent activity*.

We anticipate that this framework will hold general utility for analyzing stages and processes by which students may construct

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logical meanings rooted in their mathematical activity (Rasmussen, Zandieh, King, & Teppo, 2005), specifically reading activity. However, assuming such a pathway toward logical abstractions naturally delimits the framework's application. This is not the pathway assumed by all logic instruction, particularly instruction that bypasses students' linguistic abilities by engaging them with a formal syntax (such as truth-tables or a propositional calculus). Thus, we find it necessary to clarify our stances regarding the nature and learning goals of conditional logic for proof-oriented mathematics.

Logic can be taught using meaningful everyday statements (e.g. Epp, 2003), meaningless everyday statements, formal syntax (e.g. Hawthorne & Rasmussen, 2015), or mathematical statements (e.g. Dubinsky & Yiparaki, 2000). In this project, we only use meaningful mathematical statements quantified over some set. We do so to engage students in reading statements that are endemic to proof-oriented mathematics. We intend for students to learn to consistently use generalizable criteria to determine the truth-values of mathematical conditionals. The literature and our observations indicate that a student may affirm a quantified conditional such as "If an integer x is a multiple of 6, then x is a multiple of 3" in various ways:

1. as an empirical generalization induced from a series of examples
2. based on properties such as the spacing of these multiples on the number line
3. as the result of a proof (maybe implicitly) using the theorem "if $a|b$ and $b|c$, then $a|c$,"
4. as a subset relation between the set of multiples of 6 and the set of multiples of 3, or
5. as impossible to falsify because there is no multiple of 6 that is not a multiple of 3.

All of these constitute *meanings for the truth of a conditional* (hereafter "truth conditions"). We call these *meanings* in the sense of Thompson, Carlson, Byerley, and Hatfield (2014) who define the meaning of something as the set of inferences available from understanding it in a particular way. A truth-value is one inference that students draw from mathematical statements. We find it useful to distinguish between truth conditions that rely on the mathematical content of the sentence (examples 1–3) and those that rely on generalizable criteria (examples 3–5). We place 3 in both categories because the cited theorem is mathematically specific, but one can generalize the criterion that a conditional is true if there is a proof of the conclusions from the hypotheses (Weber & Alcock's, 2005, *warranted conditional*).

We use the term *generalizable* to acknowledge that students may or may not have considered whether their interpretation of a single statement would work on other statements. If a student uses meaning 1 above, one should not infer that this constitutes their answer to an implicit epistemic question about standards for declaring all conditionals true. This would assume without evidence that the student is engaged in reasoning about logic (Dawkins & Cook, 2017). We want to help students consider whether their criteria for truth apply viably to other conditionals, which is why we pursue student development of *conscious* truth conditions. We do not want students to avoid meanings like 2 and 3, which are mathematically essential, but they are not tantamount to understanding mathematical logic.

Our view of the goals of logic instruction for proof-oriented mathematics have been informed by a broad range of literature on logic and conditionals and our ongoing series of teaching experiments (Dawkins, 2017; Dawkins & Cook, 2017). In the following two sections, we outline the relevant contributions from those two sources to this report.

2. Situating within prior literature

The body of research on conditional statements sprawls across the literature on psychological, philosophical, and mathematics education. We situate our work within this literature using a few key distinctions. We study how students consciously learn truth conditions by reflecting on reading and assigning truth-values. The majority of philosophical literature on the truth of conditionals is prescriptive in nature, generally excluding psychological considerations as *psychologism* (Pelletier, Elio, & Hanson, 2008; Schroyens, 2010). Philosophical explorations of logics with different properties intended to capture aspects of *ideal* human reasoning; nevertheless, they aim for formal rigor and tend to ignore learning processes. The next sections relate our study to prior psychological and mathematics education studies, respectively.

2.1. Psychological studies

Psychologists have intensely investigated conditionals in their connection to inference and logic. One may describe any inference as a conditional—"if p then q" as "since I know p, I can conclude q." Psychologists like Inhelder and Piaget (1958) thus used such notation to describe children's inferences, but the inferences they studied were inferences of action and anticipation rather than assignments of truth-values to claims. They did not focus on teaching particular meanings for such inferences. The now famous studies of Wason (1966) and Johnson-Laird and Byrne (1991) are closer to our work in that they assessed how and when adults declare conditional statements true or false. However, they focused primarily on the behavior of untrained adults in either meaningless or everyday contexts. They generally found that adults do not assess everyday conditionals according to logical rules in the sense that their reasoning was sensitive to semantic content and logic is invariant across semantic content. This line of work has led some psychologists to claim that deductive reasoning has no bearing on people's everyday inferences (e.g. Schroyens, 2010), though others resist this claim (e.g. Stenning, 2002). Our work in undergraduate mathematics education is, and must be, different because proof-oriented mathematics uses some deductive norms of inference as tools for assessing acceptable proof. Unlike most psychological literature that is not beholden to our disciplinary norms, we intend for students to consciously develop truth conditions for mathematical conditionals. Dawkins and Cook's (2017) notion of *reasoning about logic* differs from two alternatives that appear most