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Definite integral registers using infinitesimals

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A R T I C L E I N F O

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A B S T R A C T

I propose a theoretical model of representation registers for definite integral notation. The two registers, adding up pieces (AUP) and multiplicatively-based summation (MBS), are developed from modes of interpreting integral notation identified by S. Jones (2015). In this model, the AUP register affords modeling with definite integral notation, while the MBS register affords sense-making with and evaluation of integrals. These registers are illustrated in the context of a Calculus I class that used an informal infinitesimals approach; in this class differentials such as d*x* directly represented infinitesimal quantities instead of serving as a reminder of a quantity that existed before a limit was taken. Theoretical implications of extending Duval's register theory (2006) in this way are also explored.

1. Introduction

I recently conducted a survey of 57 science and engineering faculty at my university, asking how important various first-semester calculus skills are for the success of students studying their discipline. The two skills they reported to be most important were the ability to interpret and model with derivatives and definite integrals.[1] When I asked one professor why these were seen to be more important than evaluating or computing integrals, he said that computers can now do all of your integrating and differentiating. But computers can't look at a situation and, say, write an integral that represents it.

If the game has been shifting for students in STEM disciplines, it is not clear that their calculus curriculum and instruction have also shifted to support this need for modeling and interpreting. The calculus reform movement advocated shift along such lines decades ago, and one of its effects is that textbooks now have more application problems (Hughes-Hallett, 2000). But are we explicitly equipping students with modes of interpreting calculus notation that support their modeling and interpreting in these application contexts? I am not talking about persistence or problem-solving strategies or other broader capacities that will support modeling by helping students untangle and make sense of complex contextual problems. I mean specifically providing students with ways of interpreting the pieces of calculus notation so that they can reliably and meaningfully represent and work with quantities in the context at hand. In this article I focus on developing a model for two such modes of interpretation for definite integral notation.

My goal is to propose and illustrate two registers that are plausibly beneficial for interpreting and working with definite integral notation, to the end that these could be used to explicitly support and interpret student reasoning in the calculus classroom. I ground these registers—adding-up-pieces (AUP) and multiplicatively-based summation (MBS)—in a course that uses an "informal infinitesimals" approach to calculus. In Section 2 I summarize prior research about student interpretations of definite integral notation, then in Section 3 I describe the informal infinitesimals approach to calculus. After this I motivate and provide a framework for registers, then in Section 6 I detail the AUP and MBS registers, after which I illustrate these registers with student examples.

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[1] The skill they reported to be least important was calculating and interpreting limits.

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2. Student modes of interpretation of definite integral

Prior research has identified several distinct ways students interpret definite integral notation. Recently, Steven Jones has investigated ways students interpret the elements of definite integral notation $\int_{[]}^{[]}$ [], with the integrand itself often in the form []d[] (Jones, 2013, 2015a, 2015b).

a) Area interpretation (or "perimeter and area") —The definite integral represents an area "under" a curve in the coordinate plane. Each box in the template is associated with part of the boundary of a shape in the plane: the limits of integration denote vertical lines for its left and right sides and the integrand forms the (usually curvy) top of the shape. The "d[]" denotes the variable on the horizontal axis, which forms the bottom of the shape. "The shape is taken as a fixed, undivided whole that is not partitioned into smaller pieces" (Jones, 2015b, p. 156Jones, 2015bJones, 2015b, p. 156).
b) Anti-derivative interpretation (the "function matching" symbolic form)—The integrand came from some other "original function" through differentiation, and now the integral symbol represents an instruction to find this original function. The differential d[] dictates the independent variable "with respect to" which the derivative had been taken, and the limits of integration are the values that one must plug into the original function to get the numerical answer. This symbolic form reflects what I will call the *anti-derivative interpretation* of definite integral notation.
c) Sum-based interpretations—These interpretations treat the integral as a sum of pieces over a specified domain. In what Jones (2015a) and Jones & Dorko (2015) called the "adding up pieces" interpretation, the domain has been broken into infinitely many infinitesimal increments of uniform size, each represented by d[]. A distinct infinitesimal piece of a sought quantity is associated with each such increment. These pieces are then summed or accumulated across the infinitely many increments in the domain to produce a total amount of that quantity. The limits of integration denote the starting and stopping places of the summation in terms of the domain variable. In (2015b), Jones began referring to this same interpretation as "multiplicatively-based summation," to highlight the multiplication in the summand, which is of the form [integrand]·d[].

In this paper I adapt Jones' terminology by distinguishing between adding up pieces (AUP) and multiplicatively-based summation (MBS) as different things, as detailed in section 6. This distinction retains the spirit of sum-based integral interpretation and Jones' findings about it, while allowing us to distinguish situations where the multiplicative structure of the summand is incidental or invisible from situations where it is crucial and salient. As I describe below in Section 6, the development of registers for AUP and MBS allows us also to see that it is not just these situations that are different, but also the purposes, interpretations, and sets of available operations.

Another sum-based interpretation is the Riemann sum interpretation, in which the definite integral is treated as the limit of a sequence of Riemann sums. Each Riemann sum is created by a particular finite partition of the domain, and the sum approximates the total quantity that the integral measures. These approximations become exact when the limit is taken. This differs from the interpretations that appeal to infinitesimals; in a limit of Riemann sums neither the "d[]" nor the "[integrand]d[]" directly assign to particular quantities, but rather are notational vestiges that call to mind quantities present in the finite Riemann sums.

In general, area and anti-derivative interpretations are very commonly displayed by calculus students, and sum-based interpretations occur more rarely (e.g., Fisher, Samuels, & Wangberg, 2016; Jones 2013, 2015a, 2015b; Marrongelle, 2001; Oehrtman, 2009; Orton, 1983; Sealey, 2006; Sealey & Oehrtman, 2005, 2007; Thompson & Silverman, 2008, Wagner,2016). For example, Jones, Lim, and Chandler (2016) surveyed 150 undergraduate students who had completed first-semester calculus, using Prompts 1 and 2 that I use in this study (see section 7.2). On Prompt 1, 87.3% of students appealed to the area interpretation; on Prompt 2, 76% used the anti-derivative interpretation. Only 22% of students made even a passing reference to summation of any kind on either prompt, and on each prompt less than 7% appealed to AUP or MB Fisher et al. (2016) found that the majority of students in a standard calculus class used only the area interpretation when describing the meaning of a definite integral, and Grundmeier, Hansen, and Sousa (2006) found that only 10% of students mentioned an infinite sum when asked to define a definite integral.

Various studies claim that sum-based interpretations of the definite integral are much more productive in general for supporting student reasoning than area and anti-derivative interpretations (e.g., Jones, 2013, 2015a, 2015b; Jones & Dorko 2015; Sealey, 2006, 2014; Sealey & Oehrtman, 2005, 2007; Thompson & Silverman, 2008; Wagner, 2016). For modeling in particular, the area and anti-derivative interpretations have serious limitations. The area interpretation is problematic when modeling in the myriad situations when the sought quantity is difficult to imagine as the area of a region (e.g., work, velocity, force, volume, arclength) (Jones, 2015a; Thompson, Byerley, & Hatfield, 2013). The anti-derivative interpretation provides even less support for modeling, since it gives only a technique for evaluating a definite integral, not for creating one (Jones, 2015a). These interpretations produce significant obstacles for students modeling with integrals in physics applications (Doughty, McLoughlin & van Kampen, 2014; Meredith & Marrongelle, 2008; Nguyen & Rebello, 2011). On the other hand, Wagner (2016) found that many upper-level physics students had developed sum-based reasoning while none of the lower-level ones he interviewed had (who had taken calculus too). This suggests that sum-based reasoning is beneficial, maybe even necessary, for students to develop in order to work with integrals in physics contexts. Jones (2013, 2015a) found that students who used sum-based reasoning were more successful on physics modeling tasks than students who used only area or antiderivative interpretations. One reason may be because sum-based reasoning focuses attention on the quantities and units of the situation being modeled. For instance, students who used MBS could appeal to the multiplicative structure between the integrand and differential to explain why they had produced the correct integrand, since (revolutions per min) · (mins) would