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Secondary mathematics teachers' meanings for measure, slope, and rate of change☆

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ABSTRACT

This article reports an investigation of 251 high school mathematics teachers' meanings for slope, measurement, and rate of change. The data was collected with a validated written instrument designed to diagnose teachers' mathematical meanings. Most teachers conveyed primarily additive and formulaic meanings for slope and rate of change on written items. Few teachers conveyed that a rate of change compares the relative sizes of changes in two quantities. Teachers' weak measurement schemes were associated with limited meanings for rate of change. Overall, the data suggests that rate of change should be a topic of targeted professional development.

1. Introduction

We agree with Copur-Gencturk (2015), Zaslavsky (1994), and Thompson (2013) that how teachers understand a mathematical idea is an important factor in the mathematical understandings that students actually form. To the extent that teachers listen to and adapt to what they understand their students to mean, teachers who understand an idea they teach coherently provide greater opportunities for students' to learn that idea coherently. Inversely, the less coherently teachers understand an idea they teach, the fewer are students' opportunities to learn that idea coherently.

Rate of change is a central idea in the secondary mathematics curriculum. It is therefore important to understand the extent to which teachers' meanings for rate of change are sufficient to support them in helping students make sense of rate of change and related ideas.

Prior studies of secondary teachers' meanings for slope, rate of change, and quotient typically focused on small numbers of teachers in an effort to model their meanings or to characterize their proficiency with these ideas (Ball, 1990; Coe, 2007; Fisher, 1988; McDiarmid & Wilson, 1991; Stump, 1999, 2001; Thompson, 1994b; Thompson & Thompson, 1994). Large scale investigations of mathematical knowledge for secondary teaching, such as the TEDS-M study of mathematical knowledge and pedagogical content knowledge did not release any items related to quotient, rate of change, fraction or measurement (Tatto et al., 2012).

Since little is known about secondary teachers' meanings for the content they teach, we developed a diagnostic instrument, *Mathematical Meanings for Teaching secondary mathematics* (MMTsm), to help researchers and professional development leaders diagnose groups of secondary teachers' mathematical meanings (Thompson, 2016). Our aim was to help professional development leaders design interventions that would address weaknesses in teachers' meanings so that teachers can better help students. Studies of elementary teachers have demonstrated that teachers' scores on assessments of Mathematical Knowledge for Teaching are related to improvement in their students' performance (Hill, Ball,

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Blunk, Goffney, & Rowan, 2007). At the same time, while the aim of the MMTsm is diagnostic, using it on a large scale allows us to examine the prevalence of particular meanings and ways of thinking in larger populations of teachers.

Our development of the rate of change items on the MMTsm was guided by qualitative work that characterized students' and teachers' thinking about rate of change, quotient, and slope (Coe, 2007; Lobato & Thanheiser, 2002; Martínez-Planell, Gaisman, & McGee, 2015; Nagle, Moore-Russo, Viglietti, & Martin, 2013; Planinic, Milin-Sipus, Katic, Susac, & Ivanjek, 2012; Stump, 1999, 2001; Thompson, 1994a, 1994b; Thompson, Carlson, Byerley, & Hatfield, 2014; Thompson & Saldanha, 2003; Thompson & Thompson, 1994; Walter & Gerson, 2007; Zaslavsky, Sela, & Leron, 2002). Construction of quality items and rubrics required articulating productive meanings for rate of change that are useful in many contexts such as calculus, science and economics. We also needed specific descriptions of common unproductive meanings for rate of change. In addition, other studies provided evidence that many secondary teachers' meanings for quotient are only productive in limited situations and suggested that investigation of teachers' meanings for "elementary" ideas is important (Ball, 1990; McDiarmid & Wilson, 1991).

This article reports 251 teachers' responses to MMTsm items that focused on high school teachers' meanings for slope and rate of change. We were interested in whether teachers' meanings for rate of change were additive, multiplicative, or both and how they coordinated additive and multiplicative meanings for slope. We examined whether teachers were able to differentiate between situations best modeled with subtraction versus division. We also were interested in the extent to which teachers' meanings for rate of change appeared to be connected to meanings for quotient as a measure of relative size.

1.1. Summary of article

1.1.1. Literature review

Identifies common meanings for rate of change, slope, measurement, and quotient that have been identified in qualitative studies and were used to write items and categorize teachers' responses.

1.1.2. Methods

Discusses the participants in study, rubric creation, and explains why open ended items with multiple acceptable answers should not be used to evaluate individual teachers.

1.1.3. Results

Includes six rate of change, slope and measurement items, associated rubrics, and teachers' responses. The results indicate that a majority of teachers' meanings for these concepts are only productive in limited circumstances.

1.1.4. Looking across items

Shows the correlation between teachers' measurement responses and multiplicative responses for rate of change and slope. Provides qualitative evidence of the limitations of chunky meanings for slope and rate of change.

1.1.5. Conclusion

Teachers need more opportunities to develop productive mathematical meanings, including meanings for slope and rate of change, in undergraduate programs and professional development.

2. Literature review

2.1. Mathematical meanings versus mathematical knowledge

Thompson and Thompson (1996) used the phrase Mathematical Knowledge for Teaching (MKT) to describe teachers' schemes for ideas they teach and which they hold at a reflected level. They described teachers' reflected schemes as guides for their interactions with students whom they hope will develop the meanings and ways of thinking that the teacher intends. Silverman and Thompson (2008) expanded this scheme-based meaning of MKT by examining how teachers might create what they called Key Pedagogical Understandings from a basis of their personal, well-formed schemes—schemes which Simon (2006) called Key Developmental Understandings. A Key Pedagogical Understanding is a mini-theory that a teacher holds regarding how to help students create the schemes that the teacher intends. In other words, A. Thompson, P. Thompson, and Silverman used *knowledge* in the sense of Piaget and von Glasersfeld—as schemes and ways of coordinating them that enable people to function adaptively in light of their goals and experienced situations. We see teachers' schemes as more than a set of declarative facts that the teachers learned about students and mathematics. An example of a declarative fact is "when students add two fractions many add the numerators and denominators." We want to model teachers' more general schemes for fraction, measure, quotient and rate of change that would allow us to predict how teachers' might respond in a large variety of situations and not just in a specific context such as teaching the procedure to add fractions.

Ball, Hill and colleagues (Ball & Bass, 2002; Hill et al., 2007; Hill, Schilling, & Ball, 2004) used the phrase MKT differently than did A. Thompson, P. Thompson, and Silverman. Ball et al. observed elementary school teachers and documented the "mathematical knowledge and skill used in the work of teaching" (Ball, Hill, & Bass, 2005, pp. 16–17). Schilling, Blunk, and Hill (2007) discussed the evolution of their understanding of what their instrument measured from "declarative knowledge" to "a kind of close reasoning":

When we began developing items in this domain, we hypothesized that teachers' knowledge of students existed separately from