Journal of Mathematical Behavior 36 (2014) 51–72

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Students' coordination of lower and higher dimensional units in the context of constructing and evaluating sums of consecutive whole numbers☆

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ARTICLE INFO

Keywords:
Units coordination
Schemes
Higher and lower dimensional units
Combinatorial reasoning
Middle grades
Models of students' reasoning
Sums of consecutive whole numbers

ABSTRACT

This paper examines how three eighth grade students coordinated lower and higher dimensional units (e.g., composite units and pairs) in the context of constructing a formula for evaluating sums of consecutive whole numbers while solving combinatorics problems (e.g., $1+2+\cdots+15=(16\times 15)/2$). The data is drawn from the beginning of an 8-month teaching experiment. The findings from the study include: (1) a framework for understanding how students coordinate lower and higher dimensional units; (2) identification of key learning that occurred as students made the transition between solving two kinds of combinatorics problems; and (3) identification of the links between the way students' coordinated lower and higher dimensional units and their evaluation of sums of consecutive whole numbers. Implications for research and teaching are considered.

Introduction

Researchers have provided mathematical analyses indicating that combinatorics problems like the Outfits and Handshake Problem have unique mathematical properties (e.g., Behr, Harel, Post, & Lesh, 1994; De Bock, Van Dooren, Janssens, & Verschaffel, 2007; Greer, 1992; Vergnaud, 1983).

> *The Outfits Problem*: You have 3 shirts and 4 pairs of pants. An outfit consists of one shirt and one pair of pants. How many possible outfits can you make?
>
> *The Handshake Problem*: There are 10 people in a room. Each person wants to shake hands with everyone. How many handshakes will there be?

One unique mathematical property these problems share is the potential to support students to create relationships among lower and higher dimensional units (Behr et al., 1994; Vergnaud, 1983). For example, as part of solving the Outfits Problem, a student can pair the first shirt with the first pants to establish an outfit where each outfit contains two units (e.g., the first shirt and first pants), but is counted as a single unit (e.g., the first outfit). This property can be thought about as involving the identity that $1^2=1$, and it is an important starting point for creating more complex relationships between

☆ The research reported in this manuscript was part of the author's doctoral dissertation completed at the University of Georgia under the direction of Leslie P. Steffe.

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http://dx.doi.org/10.1016/j.jmathb.2014.07.005

lower and higher dimensional units – in the case of the Outfits and Handshake Problem, between one and two-dimensional units.

Elsewhere I identified differences in the way that three eighth grade students coordinated lower and higher dimensional units in their solution of problems like the Outfits Problem (Tillema, 2013). In this analysis I extended Steffe's (1992, 1994) framework on units coordination by identifying novel mental operations that the students used in their solution of such problems. This paper builds on this work by: (1) illustrating the learning that occurred as the three eighth grade students made the transition from solving problems like the Outfits Problem to more advanced combinatorics problems like the Handshake Problem; and (2) identifying how differences in the way the students coordinated lower and higher dimensional units created differences in the way they constructed and evaluated sums of consecutive whole numbers, which they did as they solved the Handshake Problem and other related problems.

The data for the paper is drawn from an 8-month teaching experiment. The goal of the teaching episodes reported on in this paper was to investigate how students' coordination of lower and higher dimensional units was related to how they constructed and evaluated sums of consecutive whole numbers (e.g., $1+2+\cdots+9=(10\times 9)/2$) (see Appendix A). The broader goal of the experiment was to understand what ways of operating (including the coordination of lower and higher dimensional units) enabled and constrained the students' construction of non-linear meanings of multiplication – specifically meanings for raising a quantity to a whole number power. Therefore, I situate the paper in relation to the local goal of the teaching episodes, but in the Discussion section I connect the learning that is illustrated in the Data Analysis to the broader goal of the experiment. The research questions are:

(1) What learning occurred as students made the transition from solving problems like the Outfits Problem to problems like the Handshake Problem? (RQ1)
(2) How did students' coordination of lower and higher dimensional units contribute to differences in their construction and evaluation of sums? (RQ2)
(3) How is the learning that occurred in these teaching episodes related to the broader goal of the experiment for students to establish non-linear meanings of multiplication? (RQ3)

Literature review

Review of research on students' combinatorial reasoning

To date, researchers' investigations of students' solutions of problems like the Outfits Problem and students' solutions of problems like the Handshake Problem have largely remained distinct (see Maher, Powell, & Uptegrove, 2010 for a notable exception). That is, researchers have tended to investigate students' solutions of problems like the Outfits Problem with elementary grades students (e.g., English, 1993, 1996, 1999; Mulligan & Mitchelmore, 1997; Nunes & Bryant, 1996; Outhred, 1996), and solutions of problems like the Handshake Problem with middle grades students (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Fischbein & Gazit, 1988; Fischbein, Pampu, & Minzat, 1970; Piaget & Inhelder, 1975). However, they have not explicitly examined how students make the transition from solving problems like the Outfits Problem to solving problems like the Handshake Problem.

There have been common findings across these two bodies of literature about what contributes to students' success in this domain. These findings include that students are more successful when they are: given concrete materials (e.g., a toy bear along with shirts and pants, and asked to dress the bear) as opposed to a written statement of a problem (English, 1991, 1993; cf. Mulligan & Mitchelmore, 1997; Nunes & Bryant, 1996; Outhred, 1996); given opportunities to generate personally meaningful notations (e.g., lists and tree diagrams) (Fischbein & Gazit, 1988; Fischbein et al., 1970; Maher et al., 2010); and allowed to justify their solution process to their peers (Maher et al., 2010). These findings helped frame the design of the study, especially, how problems were presented to students, an issue I discuss in the "Methods" section.

Another relevant discussion, which is from the body of research on middle grades students' solutions of combinatorics problems, is whether middle grades students find it easier to solve combination or arrangement problems.

> *Combination problems* involve finding the number of k element sets a person can create from an n element set where the elements are not replaced, and the order of the elements does not matter (e.g., The Handshake Problem).
>
> *Arrangement problems* involve finding the number of k element sets a person can create from an n element set where each element can either appear more than one time (with replacement) or only one time (without replacement), and the order of the elements does matter (e.g., Problem 5, Appendix A).

Researchers' discussion of this issue is relevant to problems like the Handshake Problem because it can be solved as a sum without considering ordered outcomes, or it can be solved by counting ordered outcomes using multiplication (i.e., considered an arrangement problem first), and then eliminating duplicate outcomes using division. Piaget and Inhelder (1975) found that adolescents (12–15 year olds) could solve combination problems as a sum prior to solving arrangement problems. Fischbein and Gazit (1988) confirmed this finding prior to the instructional intervention in their study. After their instructional intervention, which aimed to support students to solve combination problems by counting ordered outcomes and then eliminating them, students found it considerably more difficult to solve combination problems than arrangement