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Making sense of qualitative geometry: The case of Amanda

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ABSTRACT

This article presents a case study of a seven-year-old girl named Amanda who participated in an eighteen-week teaching experiment I conducted in order to model the development of her intuitive and informal topological ideas. I designed a new dynamic geometry environment that I used in each of the episodes of the teaching experiment to elicit these conceptions and further support their development. As the study progressed, I found that Amanda developed significant and authentic forms of geometric reasoning. It is these newly identified forms of reasoning, which I refer to as "qualitative geometry," that have implications for the teaching and learning of geometry and for research into students' mathematical reasoning.

1. Introduction

Geometry in traditional elementary school classrooms has been principally about identifying canonical shapes and matching those shapes to their given names (Clements, 2004). These picture-driven geometric experiences have done little to move students beyond the stage in which they identify shapes not by their properties but by their appearance. They have argued, "That's a triangle, because it looks like a triangle." Inevitably, little conceptual change in geometry has occurred throughout the elementary grades (Lehrer & Chazan, 1998; Lehrer, Jenkins, & Osana, 1998). No new geometric knowledge is developed beyond what children already know (Thomas, 1982, as cited in Clements, 2004).

What makes learning geometry in this kind of environment especially detrimental to young children's development is that concepts of shape are stabilizing as early as age 6 (Clements, 2004). This means that if young children's engagement is not expanded beyond a set of conventional, rigid shapes, these shapes develop into a set of visual prototypes that could rule their thinking throughout their lives (Burger & Shaughnessy, 1986; Clements, 2004). For instance, most children require that triangles have horizontal bases, all triangles are acute, and one dimension of a rectangle is twice as long as the other (Clements, 2004).

Geometry does, of course, possess a significant visual component. However, as children often experience it, geometric thinking is restricted to passive observation of static images. The problem is, children don't only see shapes that way. They see them as malleable and often provide "morphing explanations" (Lehrer et al., 1998, p. 142) for shapes they identify as similar. When geometry is about static images on paper, then engagement with, and understanding of, geometry is inevitably constrained to holistic representations of those shapes. Furthermore, arbitrary attributes of shape emerge as fundamental properties when shapes are static. For example, young children distinguish between a square and a regular diamond, because they see rotation as a property of shape. A study by Lehrer et al. (1998) found that "half of the first- and

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second-grade children believed that a line oriented 50° from vertical was not straight. They characterized the line as 'slanted' or 'bent"' (p. 149).

In order to engage and develop children's various forms of geometric reasoning – where geometric reasoning is conceived as the application of geometric properties and relationships in problem solving – they must be provided with opportunities where properties of shape are made salient. An environment that has the capacity for Euclidean as well as "morphing" transformations provides the potential for such opportunities (Battista, 2001; Hölzl, 1996; Jones, 2001; Laborde, 2000). In addition, such an environment could prove to be a useful methodological tool with which to investigate children's not-necessarily-Euclidean geometric ideas and understand their development. A more elaborate framework of geometric thinking resulting from these investigations would contribute to the body of research on learners' ways of thinking about geometry, particularly for children in the elementary grades. More importantly, this framework could help teachers make connections with the various ways in which their students think about geometry.

2. Theoretical framework

The theoretical framework for this study is grounded in scheme theory and builds upon groundbreaking work by Piaget and Inhelder (1956), who carried out the first systematic investigation of children's representational thinking about the nature of space. Piaget and Inhelder found from that investigation that:

> . . . Representational thought or imagination at first appears to ignore metric and perspective relationships, proportions, etc. Consequently, it is forced to reconstruct space from the most primitive notions such as the topological relationships of proximity, separation, order, enclosure, etc., applying them to metric and projective figures yielded by perception (p. 4).

This phenomenon that the development of children's representational thought is first topological (assuming neither constant size nor constant shape), then projective (assuming constant size but not shape) and finally Euclidean (assuming both constant size and shape), has come to be referred to as the "topological primacy thesis" (Martin, 1976a).

Next I draw on von Glasersfeld's (1995) scheme theory, which is an interpretation of Piaget's theory of cognitive development (1970). A scheme is a collection of reversible mental operations that is engaged in goal-directed action. It is composed of three parts that are organized as follows: recognition of a situation; an activity associated with that situation; and an expected result of that activity (von Glasersfeld, 1995). The recognition is the result of the assimilation of an already existing scheme, which occurs when the student fits a new experience into that scheme, and the subsequent activity produces a previously experienced and therefore expected result. However, if the result is not what the student expected, there will be a perturbation. At this point the student may review the situation and modify the scheme. This modification is referred to as an accommodation. An accommodation is an act of learning. Von Glasersfeld summarizes the process as follows: "The learning theory that emerges from Piaget's work can be summarized by saying that cognitive change and learning in a specific direction take place when a scheme, instead of producing the expected result, leads to perturbation, and perturbation, in turn, to an accommodation that maintains or re-establishes equilibrium" (p. 68). In this study, I focused on a scheme of qualitative geometric alikeness that resonates with topological equivalence and which I discuss in more detail in the sections that follow.

2.1. Topology and Piaget's "topology"

It must be noted from a formal mathematical point of view that there is much about Piaget's (1956) mathematics that invites critique. His confusion of the mathematical terminology along with his use of terminology that is not mathematical is widespread (Darke, 1982; Freudenthal, 1972; Kapadia, 1974; Martin, 1976a). This makes it difficult to determine the actual sense in which his terms are used and leads us to wonder to what extent a finding of topological primacy has to do with the mathematical community's conceptions of topology. Nonetheless, it is unfortunate that critiques of Piaget's work (e.g., Kapadia, 1974; Martin, 1976a) have obscured his contributions. Replicate experiments (Esty, 1971; Laurendeau & Pinard, 1970; Lovell, 1959; Martin, 1976b) proceed from an awareness of issues with respect to Piaget's use of the formal mathematical terminology. Still, their findings do not dispute Piaget's finding of informal topological ideas in young children, only his conclusion of topological primacy.

Initial justification for analyzing the development of children's intuitive topological conceptions follows from these findings. Further justification follows from the proposition that these ideas can provide a basis for new opportunities to identify young children's specific, original, geometric ideas and to generate a space in which to engage and extend them. So, by giving emphasis to the ideas that develop rather than the order in which they develop, this investigation set out to better understand the nature of the child's representational space by modeling the development of young children's topological ideas. So as to remain open to findings of children's conceptions that lend themselves to rich and nuanced descriptions without giving in to a felt need to classify them as either topological or not, or even formal or informal, I made a "sideways move" and gave the name "Qualitative Geometry," in its broadest sense, to these findings and the attendant possibilities for future investigation and curriculum development.