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Reasoning on the complex plane via inscriptions and gesture

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ABSTRACT

Using a diagrammatic reasoning framework about inscriptions, we explored undergraduates' reasoning about complex-valued equations. Our findings suggest that reasoning geometrically requires first reasoning algebraically about algebraic inscriptions. We found students tended to create algebraic and geometric inscriptions when their verbiage could no longer support geometric reasoning. Furthermore, they incorporated similar iconic gestures for reasoning about their geometric inscriptions, which reduced to deictic gestures as they applied their previously developed reasoning to subsequent tasks. Contrary to other research, our participants' gestures did not taper off with future tasks. Rather, their gestures transformed as concepts were automatized. Moreover, our research suggests that *gestures serve as a link between verbiage and inscriptions* rather than *inscriptions serving as a link between verbiage and gesture* as other researchers claim. In promoting synchronicity of algebraic and geometric reasoning, teachers may want to capitalize on the fact that students tend to implement similar gestures as they reason.

1. Introduction

Focus in High School Mathematics: Reasoning and Sense Making (NCTM, 2009) stresses that reasoning and sense making ensures that high school "students can accurately carry out mathematical procedures, understand why those procedures work, and know how they might be used and their results interpreted" (p. 3). It is difficult to tease out the differences between reasoning and sense making, but the authors of this document define *reasoning* as "the process of drawing conclusions on the basis of evidence or stated assumptions" (p. 4), which we adopted for our research. Our focus is not on sense making, but these notions are intertwined from informal observations based on empirical evidence to formal inferences based on logic. Our interest lies at the intersection of algebraic and geometric reasoning, specifically as it relates to complex-valued equations.

We borrow Carraher and Schliemann's (2007) definition of algebraic reasoning, which is a "process involved in solving problems that mathematicians can easily express using algebraic notation" (p. 670). We also borrow Battista's (2007) definition of geometric reasoning, which consists "of the invention and use of formal *conceptual systems* to investigate shape and space" (p. 843). These definitions complement NCTM's (2009) characteristics of reasoning with algebra and geometry. NCTM, like Carraher and Schliemann, includes arithmetic within algebra and offers the following characteristics for algebraic reasoning: meaningful use of symbols, mindful manipulation of equations, reasoned solving, connecting algebra with geometry, and linking expressions and functions. NCTM offers the following characteristics for geometric reasoning: conjecturing

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about geometric objects, constructing and evaluating geometric arguments, multiple geometric approaches, and geometric connections to and modeling with algebra. Like Battista's definition, these elements encompass spatial reasoning.

While the NCTM (2009) document addresses high school mathematics, the wealth of research exploring students' algebraic and geometric reasoning in algebra, calculus, linear algebra, differential equations, analysis, and abstract algebra (Gibson, 1998; Sierpinska, 2000; Tabaghi & Sinclair, 2013; Tall & Vinner, 1981; Zaskis, Dubinsky, & Dautermann, 1996) highlights the importance of developing students' reasoning at the collegiate level. One content domain, which has not received much attention, is complex variables. The purpose of our research is to contribute to the literature on algebraic and geometric reasoning, but about complex variables. Our overarching research question is: What is the nature of students' geometric and algebraic reasoning about complex-valued equations? Specifically, we explore the research question: what is the nature of our participants' use of diagrams, algebraic symbolism, and gesture as part of their algebraic and geometric reasoning about complex-valued equations? This research suggests that as participants explained their geometric reasoning of complex-valued equations they: (1) navigated less between algebraic and geometric reasoning as the tasks progressed, although they tended to reason with algebraic inscriptions first (2) created inscriptions when their verbiage could no longer support geometric reasoning, and (3) incorporated iconic gestures for reasoning about their geometric inscriptions which reduced to deictic gestures as they applied previously developed reasoning.

2. Literature review

We commence the literature review with an overview of research on complex numbers followed by literature on inscriptions, and finally literature related to the use of gesture. We conclude by summarizing the relationship between reasoning, inscriptions, and gesture.

2.1. Research on complex numbers

Researchers (Danenhower, 2006; Harel, 2013; Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012; Panaoura, Elia, Gagatsis, & Giatilis, 2006) have recently begun to explore the teaching and learning of complex numbers and variables. In an effort to explore undergraduates' understanding of complex numbers, Danenhower investigated their ability to navigate between different forms of complex numbers. Danenhower asked his participants to convert various instantiations of the form $(a+ib)/(c+id)$ to either the Cartesian form, $a+ib$, or the exponential form, $re^{i\theta}$. (Danenhower refers to this as the polar form.) He found that students could shift between Cartesian and exponential forms, but they did not view each one as part of a coherent whole. Furthermore, "nearly half [the participants] did not have good judgment about when to shift to another form" (p. 151). The students were unable to utilize geometric reasoning and the exponential form in a single unified approach. Instead, they turned to geometric reasoning only as an aid to the trigonometric calculations that arise from the exponential form (i.e., $re^{i\theta} = r\cos\theta + r\sin\theta$). Danenhower's findings show that students relied on algebraic approaches and only employed diagrams to validate their calculations.

In contrast to Danenhower's (2006) focus on students' navigation between *forms* of complex numbers, Panaoura et al. (2006) explored high school students' fluency in navigating between algebraic and geometric *representations* of tasks involving equations and inequalities of complex numbers. The researchers administered two questionnaires; the first questionnaire asked participants to convert an algebraic representation into a corresponding geometric representation, and the second questionnaire assessed the reverse direction. Panaoura et al. found that "the geometric approach was used more frequently, while the pupils used the algebraic approach more consistently and in a more persistent way" (p. 681). That is, these participants switched to geometry more commonly than they switched to algebra, but once inside an algebraic mode of reasoning they tended to stay there for longer stretches of time compared to geometric reasoning.

Instead of investigating how students reason about complex numbers and complex-valued functions, other researchers such as Harel (2013) and Nemirovsky et al. (2012) have attempted to help develop students' reasoning of complex numbers via teaching experiments. For example, using the DNR framework, Harel implemented a curricular unit with in-service and pre-service teachers on complex numbers that followed a historical account of the development of complex numbers. A finding of his work, consistent with our results, is that the research participants tended to reduce tasks to something that they recognized before continuing with the tasks. Following an embodied cognition philosophy, Nemirovsky et al. incorporated physical models to develop pre-service teachers' "geometric interpretation" of the product of two complex numbers. The classroom "floor tile" served as the complex plane and string and stick-on dots served as vectors or points on the complex plane. By physically moving these objects or themselves around the complex plane, the research participants reasoned that multiplying by i corresponds to a rigid 90° rotation of the entire complex plane, about the origin. In addition to the "embodied" complex plane, students calculated algebraic equations corresponding to their physical actions to test and corroborate their results.

Contradicting previous literature (Danenhower, 2006; Panaoura et al., 2006; Tall & Vinner, 1981), Nemirovsky et al. (2012) found that students acknowledged when their corresponding algebraic computations and embodied reasoning disagreed. In particular, a student who believed that multiplication by i should reflect a point across the imaginary axis in the embodied plane showed surprise when the corresponding algebraic calculation yielded a point in an unexpected location. Through further algebraic calculations and experimentation with the embodied complex plane, the student formulated correct reasoning regarding complex number multiplication. Nemirovsky et al. therefore postulate that this unexpected result led their