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Functions via everyday actions: Support or obstacle?

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ABSTRACT

The general context of this paper is the power of intuitive thinking, and how it can help or hinder analytical thinking. The research literature in cognitive psychology teems with tasks where intuitive thinking leads subjects to "non-normative" answers, including tasks for which they have all the knowledge necessary for the normative answer. The best explanation to date for such phenomena is dual-process theory, which stipulates the activation of a quick automatic *intuitive* process (*System 1*), together with the failure of the heavy, lazy, and computationally expensive *analytical* process (*System 2*) to intervene and correct the intuitive response.

In an earlier paper, we have documented a clash between intuitive and analytical thinking concerning functions, which we have termed the *changing-the-input* phenomenon. The discovery of the changing-the-input phenomenon, however, left us with a puzzle: Why has this phenomenon concerning functions – a purely mathematical concept – been observed in computer science classes but not in mathematics ones? The purpose of the present paper is to address this puzzle. More generally we ask, under what conditions the changing-the-input phenomenon will or will not be manifested? Still more generally, in learning about functions, when is the intuitive scaffolding of functions via actions-on-tangible-objects helpful, and when does it get in the way of deeper understanding?

1. Introduction

The general context of this paper is the power of intuitive thinking, and how it can help or hinder analytical thinking (Leron & Hazzan, 2006, 2009; Paz & Leron, 2009). The research literature in cognitive psychology teems with tasks where intuitive thinking leads subjects to "non-normative" answers, including tasks for which they have all the knowledge necessary for the normative answer (Kahneman, 2002, 2011). The best explanation to date for such phenomena is dual-process theory, which stipulates the activation of a quick automatic *intuitive* process (*System 1*), together with the failure of the heavy, lazy,[1] and computationally expensive *analytical* process (*System 2*) to intervene and correct the intuitive response. (For state-of-the-art discussion of dual-process theory and some of the debate surrounding it, see Evans & Frankish, 2009; Evans & Stanovich, 2013.)

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[1] We follow Kahneman (2011) and others in the dual-process literature (as well as Dawkins' "selfish gene") in using anthropomorphic metaphors in the discussion of systems 1 and 2. The assumption behind such use is that it improves the readability of the text, while anyone who would want to explicate the metaphor in more precise terms could easily do it.

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A lesser-known phenomenon, also well explained by dual-process theory, is that harder or messier input may sometime *enhance* subjects' performance (Kahneman, 2011, p. 65; Song & Schwarz, 2008). The explanation according to dual-process theory is that an easy input promotes automatic system 1 reaction, whereas a messier input generates a more effortful reaction, which may suppress intuitive response and promote analytical approach, hence the better performance. In this paper, we continue our investigation of functions by documenting a similar phenomenon in a mathematical context, where a more intuitive task formulation, which would be expected to be easier, actually results under certain conditions in less successful performance.

Specifically, in an earlier paper (Paz & Leron, 2009, henceforth abbreviated P&L) we have documented a clash between intuitive and analytical thinking concerning functions, which we have termed the *changing-the-input* phenomenon (see next section). This discovery, however, left us with a puzzle: why has this phenomenon concerning functions – a purely mathematical concept – been observed in computer science classes but not in mathematics ones? The purpose of the present paper is to address this puzzle. More generally, we ask, under what conditions the changing-the-input phenomenon will or will not be manifested? Still more generally, in learning about functions, when is the intuitive scaffolding of functions via actions-on-tangible-objects helpful, and when does it get in the way of deeper understanding?

2. Theoretical preliminaries

2.1. "Does a function change its input?"

When this question is formulated precisely and put to empirical test, it turns out that the mathematical answer is that it does not, but in some situations, many students think that it does.

It is well known that learning and understanding functions is beset with many difficulties and misconceptions (DeMarois & Tall, 1999; Eisenberg, 1991; Harel & Dubinsky, 1992; Leinhardt et al., 1990; Sierpinska, 1992; Vinner, 1983). In an attempt to make the notion of function more intuitive, teachers often invoke the image of function as action on concrete objects.[2] For example, "the function that multiplies a number by 2", "the function that removes the first element of a sequence".[3] (Following Wilensky, 1991, the term "concrete" here means that for these students, these objects are familiar and easy to imagine and to mentally manipulate.)

At the beginning of group theory courses, lecturers often use operations on everyday objects as example for general functions and their composition. For example, suppose you want a concrete metaphor for the proposition that the inverse of the product of group elements equals the product of the inverses in reverse order: $(g{\cdot}h)^{-1} = h^{-1}{\cdot}g^{-1}$. Then you may for example take g to be the operation *putting on your shoes* and h the operation *putting on your socks*, then $g{\cdot}h$ is the operation of first putting on your socks and then your shoes (in that order). To *undo* this composite operation (i.e., perform $(g{\cdot}h)^{-1}$) you need first to take off your shoes (g^{-1}), then to take off your socks (h^{-1}) (note the reverse order).[4]

P&L have demonstrated that thinking of functions as actions on objects may be a mixed blessing. While this image may indeed help introduce functions and their composition in a natural, intuitive way, we have documented that later on, the same image can lead to what we have called the *changing-the-input* misconception. To recap, consider the function F which "removes the first element" of any number sequence S, so that $F(S)$ is the sequence S without its first element. For example, if $S = (1, 2, 3, 4, 5)$ then $F(S) = (2, 3, 4, 5)$. P&L have documented that students often think that after you have applied the function F to the sequence S, the value of S has changed and is now (2, 3, 4, 5); the output of the function, they believe, has been substituted into the input variable. After all, if the function F "removes the first element" of the sequence $S = (1, 2, 3, 4, 5)$, it is natural to expect that after the operation of the function, the first element has indeed been removed and the sequence S will have then become (2, 3, 4, 5). Formally, this belief represents a misconception, since functions merely *map* their input to their output, while leaving the value of the input variable unchanged.

In P& L we have also introduced the theoretical construct of the *actions-on-objects scheme*, and have used it to explain the changing-the-input misconception. We have argued that this misconception could be explained by the Piagetian mental scheme of action on objects – part of everyone's natural thinking – where the basic image is that when we perform some action on an object, *the object has changed but has still remained the same object* (Piaget, 1983/1970). The actions-on-objects scheme fits our daily experience well: when for example we squeeze a clay ball, the ball changes its shape, but still remains for us "the same" ball.

Applying the actions-on-objects scheme to functions, however, may sometimes clash with their mathematical definition where, historically, the original intuitive language of process has been replaced by static formal algebraic formulation (cf. "Calculus without space or motion" in Lakoff & Núñez, 2000, chapter 14). As has been documented in P&L, what had been natural thinking in early stages of development has become detrimental to learning the formal concept of function.

[2] Most of the functions in the school curriculum are from the real numbers to the real numbers and can be represented by a graph, but we are concerned here mainly with a more general notion of function, which is prevalent in higher mathematics, and often goes under the names of operation, transformation, permutation, and the like.

[3] As we later explain, it is not formally correct to say that a function "multiplies a number" or "removes an element".

[4] Note that this popular analogy can serve as a metaphor but not as a legitimate "intuitive explanation", because there is no way these operations can be formally modeled as group elements (e.g., what would g^2 be in such a group?).