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Do students confuse dimensionality and "directionality"?

Ceneida Fernández[a,*], Dirk De Bock[b,1], Lieven Verschaffel[b,2], Wim Van Dooren[b,3]

[a] University of Alicante, Spain
[b] University of Leuven, Belgium

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ABSTRACT

The aim of this research is to understand the way in which students struggle with the distinction between dimensionality and "directionality" and if this type of potential confusion could be a factor affecting students' tendency toward improper linear reasoning in the context of the relations between length and area of geometrical figures. 131 9th grade students were confronted with a multiple-choice test consisting of six problems related to the perimeter or the area of an enlarged geometrical figure, then some interviews were carried out to obtain qualitative data in relation to students' reasoning. Results indicate that more than one fifth of the students' answers could be characterized as based on directional thinking, suggesting that students struggled with the distinction between dimensionality and "directionality". A single arrow showing one direction (image provided to the students) seemed to strengthen the tendency toward improper linear reasoning for the area problems. Two arrows showing two directions helped students to see a quadratic relation for the area problems.

1. Introduction

We report a study aimed at expanding our scientific knowledge on a systematic error that is committed by numerous students of a variety of ages in very diverse mathematical and scientific domains, namely the unwarranted application of proportionality or linearity in non-linear problem situations, a tendency sometimes referred to as the "illusion of linearity" (Van Dooren, De Bock, Janssens, & Verschaffel, 2008). Research has shown that this tendency is persistent to change by instruction. Van Dooren, De Bock, Hessels, Janssens, and Verschaffel (2004) developed and implemented a lesson series with the aim to break 9th graders' tendency to give linear responses in non-linear situations, more specifically in the context of the relationships between the linear measures of a figure and its perimeter, area and volume. They found that non-linear

* Corresponding author at: Departamento de Innovación y Formación Didáctica, Universidad de Alicante, Ap. de Correos, 99., 03080 Alicante, Spain. Tel.: +34 965903400x2028; fax: +34 965909700.

E-mail addresses: Ceneida.fernandez@ua.es (C. Fernández), Dirk.debock@kuleuven.be (D. De Bock), Lieven.verschaffel@ppw.kuleuven.be (L. Verschaffel), Wim.vandooren@ppw.kuleuven.be (W. Van Dooren).

1 KU Leuven, Faculty of Economics and Business, Center for Business Management Research, Warmoesberg 26, B-1000 Brussels, Belgium. Tel: +32 16604250.

2 KU Leuven, Faculty of Psychology and Educational Sciences, Center for Instructional Psychology and Technology, Dekenstraat 2, Box 3773, B-3000 Leuven, Belgium. Tel: +32 16326258.

3 KU Leuven, Faculty of Psychology and Educational Sciences, Center for Instructional Psychology and Technology, Dekenstraat 2, Box 3773, B-3000 Leuven, Belgium. Tel: +3 216325755.

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relations and the effect of enlargements on area and volume remained intrinsically difficult and counterintuitive for many students even after extensive instructional attention.

However, in the same study, it was shown that students who, by the end of the lesson series, finally understood that the length–area relationship is quadratic, suddenly started to doubt about the nature of the linear length–perimeter relationship. The authors exemplified this with a striking question raised by a student in the final lesson:

> "I really do understand now why the area of a square increases 9 times if the sides are tripled in length, since the enlargement of the area goes in two dimensions. But suddenly I start to wonder why this does not hold for the perimeter. The perimeter also increases in two directions, doesn't it?" (Van Dooren et al., 2004, p. 496).

This quote suggests that the student struggled with the distinction between dimensionality and "directionality" (the perimeter of a square is one-dimensional, but it has two "directions"). Of course, "directionality" is not a genuine mathematical term, but we use it for referring to the different directions a geometrical (plane) figure can have. For example, a triangle has three directions, a square has two directions (if we assume that parallel sides have the same direction), and a regular pentagon has five directions. In this study we investigate if the potential confusion between dimensionality and "directionality" could be a factor affecting students' tendency toward improper linear reasoning.

In the next section we first present an overview of the literature on the illusion of linearity. We focus on the domain of geometry where previous studies have evidenced that students tend to treat the relations between length and area or between length and volume as linear instead of, respectively, quadratic and cubic. Second, we summarize the scarce literature on the concept of dimensionality since it plays an important role in the principle governing the relation between the perimeter and the area of an enlarged or reduced geometrical figure. Although studies on this concept are rare, it is shown that students commonly struggle with understanding dimensionality.

2. Theoretical and empirical background

2.1. The illusion of linearity

Linearity is a powerful tool to model real-life situations, even if these situations are only approximately linear. For that reason, one major goal of mathematics education at all levels is to obtain both procedural fluency and conceptual understanding of linearity in its variety of forms and applications (Cramer, Post, & Currier, 1993; Kalchman & Koedinger, 2005). However, the educational attention that goes to linearity at numerous occasions in students' school careers, along with the intrinsically simple and intuitive nature of the linear model (Rouche, 1989), has a serious drawback: It may lead to a tendency in students to see and apply linearity anywhere, thus also in situations that are not linear at all. Already in 1983, Freudenthal warned for that pitfall: "*Linearity is such a suggestive property of relations that one readily yields to the seduction to deal with each numerical relation as though it were linear*" (p. 267).

Examples of the misuse of linearity can be found at different age levels and in various mathematical and scientific domains (Fernández, Llinares, Van Dooren, De Bock, & Verschaffel, 2012; for a review, see Van Dooren et al., 2008). For instance, in a study on arithmetic word problems solving by Cramer et al. (1993), 32 out of 33 pre-service elementary school teachers answered by means of a proportion ($9/3 = x/15$) to the problem "*Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?*" More recently, Van Dooren, De Bock, Hessels, Janssens, and Verschaffel (2005) observed that Flemish primary school pupils' performance on linear word problems considerably improved from 3rd to 6th grade. But they also observed that during the same period, pupils' tendency to over-use linear methods to non-linear problems increased accordingly: Whereas in 3rd grade 30% of all non-linear problems were solved linearly, this percentage increased until 51% in 6th grade. This tendency has been confirmed by Fernández, Llinares, Van Dooren, De Bock, and Verschaffel (2011) with Spanish secondary school students: Whereas 7th and 8th grade students were more successful in solving non-linear problems, 9th and 10th grade students were more successful in solving linear ones.

Furthermore, a (re-)analysis of both well-known and less-known probabilistic misconceptions by Van Dooren, De Bock, Depaepe, Janssens, and Verschaffel (2003b) showed that these are often interpretable in terms of improper applications of linearity. A typical example is students' belief that the probability of at least one success in a game of chance is directly proportional to the number of trials. In the domain of calculus, an example of the over-use of linearity related to university students was provided by Esteley, Villarreal, and Alagia (2004): 62% of students involved in a first calculus course responded linearly with respect to the increase of the height as a function of time, instead of taking into account the exponential character of this growth process in the problem: "*If a plant measures 30 cm at the beginning of an experiment, and its height increases 50% monthly, how much will it measure after 3 months?*".

This paper focuses on students' misuse of linearity in geometry. In this domain, one of the best-known and most frequently investigated cases relates to application problems about the effect of an enlargement or reduction of a figure on its area or volume. The principle governing this type of problems is that an enlargement or reduction with factor k enlarges all lengths (and thus also the perimeter) with factor k, the area with factor k^2, and—for a solid—the volume with factor k^3. A crucial aspect in understanding this principle is the insight that these factors only depend on the dimensions of the magnitudes involved (length, area, and/or volume) and not on the type of figure (square, triangle, circle, cube,. . .). When the sides of a triangle are tripled, the perimeter of the triangle is tripled too, but its area is multiplied by 9. According to Freudenthal