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Towards a mathematically more correct understanding of rational numbers: A longitudinal study with upper elementary school learners

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ABSTRACT

In this study we longitudinally followed 201 upper elementary school learners in the crucial years of acquiring rational number understanding. Using latent transition analysis we investigated their conceptual change from an initial natural number based concept of a rational number towards a mathematically more correct one by characterizing the various intermediate states learners go through. Results showed that learners first develop an understanding of decimal numbers before they have an increased understanding of fractions. We also found that a first step in learners' rational number understanding is an increased understanding of the numerical size of rational numbers. Further, only a limited number of learners fully understand the dense structure of rational numbers at the end of elementary education.

1. Introduction

There is broad agreement in the literature that a good understanding of the rational number domain is highly predictive for the learning of more advanced mathematics (e.g., Siegler, Thompson, & Schneider, 2011). It is therefore worrying that many elementary and secondary school learners and even (prospective) teachers face serious difficulties understanding rational numbers. For instance, Van Hoof, Verschaffel, and Van Dooren (2015) gave the following problem to a representative group of 4th, 6th and 8th graders: “What is half of 1/8?”. Only 8% of the 4th, 47% of the 6th, and 63% of the 8th graders could accurately answer this question. Further, a survey of a national representative sample of American algebra teachers showed that a lack of rational number understanding is one of the major sources why learners are not performing well in algebra classes (Hoffer, Venkataraman, Hedberg, & Shagle, 2007). Finally, based on their review of 43 studies from all over the world on prospective teachers' rational number understanding, Olanoff, Lo, and Tobias (2014), concluded that most prospective teachers are accurate in performing procedures with rational numbers, but struggle to understand the meanings behind the procedures and the reasons why the procedures work.

An often reported source for the struggle with understanding rational numbers is the natural number bias, i.e., the tendency to (inappropriately) apply properties of natural numbers in rational numbers tasks (e.g., Behr, Lesh, Post, & Silver, 1983; Gómez, Jiménez, Bobadilla, Reyes, & Dartnell, 2014; Ni & Zhou, 2005; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Vamvakoussi, Van Dooren, & Verschaffel, 2012, Vamvakoussi & Vosniadou, 2004, 2010; Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015; Vosniadou, 2013).

The literature reports at least three aspects of the natural number bias, relating to size, operations, and density. The first aspect involves the numerical size of numbers. Learners often consider a fraction as two independent numbers, instead of a ratio between the numerator and denominator. This incorrect interpretation of a fraction can lead to the misconception that the numerical value of a fraction increases when the numerator, denominator, or both increase, just like it is the case with natural numbers (e.g., McMullen, Laakkonen, Hannula-Sormunen, & Lehtinen, 2015). For example, 1/8 can be judged larger than 1/6, just like 8 is larger than 6. Similarly, in the case of decimal numbers, some learners have been found to wrongly assume that, just like it is the case with natural numbers, longer decimals are larger, while shorter decimals are smaller. For example, these learners judge 0.12 larger than 0.8, just like 12 is larger than 8 (e.g., Meert, Grégoire, & Noël, 2010a, 2010b; Stafylidou & Vosniadou, 2004).

The second aspect concerns the effect of arithmetic operations. After learners did arithmetic with mostly natural numbers only in their first years of schooling, some learners have been found to apply the rules that hold for natural numbers also to rational numbers, also in cases where this is inappropriate. These group of learners assume for example that addition and multiplication will lead to a larger result, while

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subtraction and division will lead to a smaller result. For example, learners think that 5 ∗ 0.32 will result in an outcome larger than 5 (e.g., Christou, 2015; Van Hoof et al., 2015).

The third aspect is density. Many researchers (e.g., Merenluoto & Lehtinen, 2004; Vamvakoussi, Christou, Mertens, & Van Dooren, 2011; Vamvakoussi et al., 2012, Vamvakoussi & Vosniadou, 2004, 2010; Van Hoof et al., 2015; Vosniadou, 2013) reported a lack of understanding of the dense structure of rational numbers. Contrary to natural numbers that have a discrete structure (each natural number has a successor number; after 5 comes 6, after 6 comes 7, …), rational numbers are densely ordered (between any two rational numbers are always infinitely many other numbers). This difference in structure of both types of numbers leads to frequently found mistakes such as thinking that there are no numbers between two pseudo-successive numbers (e.g., 6.2 and 6.3 or 2/4 and 3/4 (e.g., Merenluoto & Lehtinen, 2004; Vamvakoussi et al., 2011)).

Evidence for this natural number bias has been frequently found in the much higher accuracy levels of learners on congruent rational number tasks (i.e., tasks where natural number reasoning leads to the correct answer, for example: "Which number is the larger one: 0.45 or 0.2?"), compared to their accuracy levels on incongruent tasks (i.e., tasks where natural number reasoning leads to the incorrect solution, for example: "Which number is the larger one: 0.45 or 0.6?").

A lot of research on learners' transition from natural to rational numbers has been described from a conceptual change perspective (but see for example Ni & Zhou, 2005 for alternative views on the origin of the natural number bias). This conceptual change perspective argues that since children encounter natural numbers much more frequently than rational numbers in daily life and in the first years of instruction, they form an idea of what numbers are and how they should behave based on these first experiences with and knowledge of natural numbers. For instance, they think that numbers are discrete, that they "get bigger" with addition or multiplication while subtraction or division makes them "smaller", etc. So, to overcome the natural number bias, a conceptual change revising these initial natural number based understandings is required once rational numbers are introduced in the classroom (e.g., McMullen, 2014; Vamvakoussi & Vosniadou, 2004, 2010; Van Hoof et al., 2015; Vosniadou & Verschaffel, 2004). It should be noted that in the conceptual change literature, there is an ongoing debate on whether learners' initial ideas of concepts are to be characterized as relatively independent fragments (e.g., diSessa, 2013) or as a more or less coherent theory (e.g. Vosniadou, 2013). Nonetheless, in both views conceptual change is considered to be not an all or nothing issue but a gradual and time-consuming process, with many intermediate states between the initial and the correct understanding (e.g., Vosniadou, 2013). Vosniadou (2013) goes further by defining a special class of intermediate states, which she calls "synthetic conceptions". They refer to combinations of elements of the initial idea of number with elements of the new information assimilated in the knowledge structure. An illustration is the synthetic conception of rational numbers as being a collection of unrelated sets of numbers based on their representation (i.e., natural numbers, fractions, and decimal numbers are three unrelated sets of numbers) that are allowed to have different properties. For example, some learners think that there are infinitely many decimal numbers between two decimal numbers, while at the same time they do not accept that there can be infinitely many fractions between two given fractions (Vamvakoussi & Vosniadou, 2010). Therefore, it seems important to investigate in detail how the process of conceptual change occurs from the initial natural number based idea of a rational number towards a mathematically more correct one; which correct insights are gained first, and to characterize the intermediate states that can be found in learners. More specifically, it is essential to investigate whether general patterns in this development can be found. If so, a learner's profile at a certain measurement point can be considered to be predictive for its further development. From an educational perspective, such profiles would be helpful for teachers to provide effective instruction that is adapted to the specific knowledge and needs of each learner (Schneider & Hardy, 2013).

While the natural number bias has already generated substantial research interest in the last decade, empirical evidence on the development of learners' conceptual change in the longer term is scarce. Moreover, studies that try to uncover that development are typically cross-sectional (e.g., Stafylidou & Vosniadou, 2004; Van Hoof et al., 2015). The single exception we are aware of is the recent longitudinal study of McMullen et al. (2015), wherein 263 upper elementary school children have been followed over a one-year time period, including two different school years. The researchers measured children's conceptual understanding of the numerical size and the dense structure of rational numbers. Although the developmental patterns that were found indicated that only a limited number of children showed conceptual change at all in both aspects, it was concluded that a good understanding of the numerical size of rational numbers is a necessary but not a sufficient step for a good understanding of the dense structure of rational numbers. Our study builds on this first longitudinal study, but extends it in several ways.

1.1. The present study

Previous studies stated that many learners need a conceptual change, characterized by several intermediate states and possible synthetic models, in order to come to a good understanding of the rational number domain (see above). However, it remains unclear what these intermediate states consist of and whether there is some consistency in these states across students and across educational systems.

In the present study, we will longitudinally follow the development of rational number understanding of upper elementary school learners in the crucial years of acquiring rational number understanding. The aim of this study is to have a theoretical contribution to the research field by empirically characterizing in detail the intermediate states of learners' conceptual change from an initial natural number based concept of rational numbers towards a mathematically more correct one and by investigating whether these intermediate states have a consistent character across students or not.

Next to the general goal of characterizing the intermediate states, we also want to shed light on three important aspects of learners' rational number development, by extending previous research, and particularly the longitudinal study by McMullen et al. (2015).

First, the results of this study will allow us to take a glance at the question to what extent the development of learners' understanding of rational numbers depends on the kind of rational number instruction that is given (for a broader discussion on cross-national differences in rational number knowledge, see Nguyen, 2015). In Finland, the country where the study of McMullen et al. (2015) took place, rational number instruction is given in intensive periods where the focus in the mathematics class is for a few weeks (mostly) only on rational numbers, whereas in Flanders rational number instruction is spread out throughout several years. By comparing the results of the study of the Finish learners of McMullen et al. (2015) with the results of our study, we have some indications whether the same developmental patterns can be found despite a different curricular approach. Second, keeping in mind that different sorts of misconceptions are found in decimal versus fraction tasks (e.g., Resnick et al., 1989), it is possible that learners' understanding of these two representation types develops differently. This possibility was not yet systematically taken into account in previous research. Our study will allow to address this specifically. Third, while McMullen et al. (2015) concluded that a good understanding of the aspect of size forms a prerequisite to understand the dense structure of rational numbers, the third aspect that we distinguished above, i.e. operations, was not considered in that study. Thus, it remains unclear how learners' understanding of the aspect of operations develops as compared to size and density understanding. Starting from previous research based on the conceptual change perspective (e.g.,