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Adolescents' understanding of inversion and associativity☆

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ABSTRACT

The present study investigated when an understanding of the mathematical concepts of inversion and associativity matures and whether the application of these concepts to problem solving requires the interruption of computational strategies (e.g., Siegler & Araya, 2005). In the study, 40 adolescent participants per grade from Grades 7, 9, and 11 and 40 adult participants solved multiplication and division inversion and associativity problems and completed a task that measured whether the execution of the inversion shortcut or associativity strategy prevents the execution of competing computational strategies. Inversion shortcut use approached adult levels in Grade 9. Associativity strategy use significantly increase in early adulthood. Also, there was considerable individual variability in strategy use. Finally, the execution of both conceptually-based strategies interrupted computational strategies. Thus, adolescence is an important developmental period for understanding multiplicative concepts and applying conceptual mathematical knowledge to problem solving may require the interruption of procedural knowledge.

Researchers have used mathematical problem solving to study how problem solving strategies are selected, how new problem solving strategies develop, and what factors affect problem solving. As a result, detailed models of strategy choice and development have been proposed (e.g., Shrager & Siegler, 1998; Siegler & Lemaire, 1997). However, current models do not adequately explain how deep, abstract knowledge (i.e., conceptual knowledge) is applied to problem solving. The present study contributes to this literature by investigating the development of two mathematical concepts across adolescence—a developmental period largely ignored by mathematical cognition researchers—and provides an empirical test of Siegler and Araya's (2005) SCADS* model of how conceptual knowledge is applied to problem solving.

1. Conceptually-based strategy use

Research investigating the development of the inversion and associativity concepts has provided some of the most detailed information on the development of mathematical problem solving that requires a deep, abstract, and flexible understanding of mathematics (Robinson & LeFevre, 2012). Individuals who understand the inversion concept know that adding and subtracting or multiplying and dividing a number by the same number results in no change to the original number, $a + b - b = a$, $d \times e \div e = d$ (Robinson & Ninowski, 2003; Starkey & Gelman, 1982). Individuals who understand the associativity concept know that numbers can be decomposed and recombined in various ways and result in same answer, $(a + b) - c = a + (b - c)$, $(d \times e) \div f = d \times (e \div f)$ (Canobi, Reeve, & Pattison, 1998; Robinson, Ninowski, & Gray, 2006). Having an understanding of the concepts of inversion and associativity is important for developing an understanding of other mathematical concepts such as commutativity, additive composition, and related complement (Baroody, Torbeyns, & Verschaffel, 2009); developing an understanding of the relationship between operations; and can result in the use of novel conceptually-based strategies (Bryant, Christie, & Rendu, 1999; Robinson & Dubé, 2009a). Individuals' understanding of inversion and associativity can be studied by determining whether an individual can apply the concepts to problem solving, producing two fast and accurate conceptually-based strategies called the inversion shortcut (e.g., $3 \cancel{\times 24 \div 24} = 3$) and the associativity strategy (e.g., $4 \times 24 \div 8$; $24 \div 8 = 3$, $3 \times 4 = 12$[1]; Rasmussen, Ho, & Bisanz, 2003; Robinson et al., 2006; Siegler & Stern, 1998).

1.1. The inversion shortcut

Inversion shortcut use has been studied across childhood and adulthood but not in adolescence. Researchers studying preschool children's understanding of the inversion concept have found that, on average, children between three and four years of age can use the inversion shortcut to solve inversion problems involving physical items (e.g., blocks, Bryant

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[1] This notation is read as follows: problem: step 1, step 2, … = answer.

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et al., 1999; Sherman & Bisanz, 2007). After the onset of formal schooling, children's inversion shortcut use is stable across Grades 2 through 4 with the shortcut being applied to just under half of three-term addition and subtraction inversion problems (Bisanz & LeFevre, 1990; Robinson & Dubé, 2009a). This suggests that Grade 4 children's understanding of the inversion concept is not markedly different from that of Grade 2 children. In Grades 6 and 8, children use the inversion shortcut less frequently on multiplication and division problems (28%) than on addition and subtraction problems (49%) and they do not exhibit any crossover (i.e., inversion shortcut use on addition and subtraction problems does not promote inversion shortcut use on multiplication and division problems; Robinson et al., 2006). Also, children's inversion shortcut use on multiplication and division problems is stable across Grades 6 through 8 (Robinson & Dubé, 2009b). This suggests that children's inversion shortcut use on multiplication and division problems does not increase across late childhood. Importantly, children do not simply switch from not using the inversion shortcut to frequent use. Close to half of the children use an intermediate problem solving method immediately preceding the discovery of the inversion shortcut (i.e., the negation strategy; e.g., $3 + 6 - 6$: $3 + 6 = 9$, $3 \cancel{+ 6 - 6} = 3$; Robinson & Dubé, 2009c; Siegler & Stern, 1998). By adulthood, the inversion shortcut is used on the majority of inversion problems, regardless of the operations involved (Robinson & Ninowski, 2003). Thus, understanding of the inversion concept begins to form well before the onset of formal schooling but maturation of this understanding is incremental and prolonged. Researchers have not yet identified when individuals' understanding of the inversion concept reaches adult levels. Given the considerable amount of research asking whether children can use the inversion shortcut—much of which is predicated on the premise that the inversion concept is essential for later mathematical knowledge (Baroody et al., 2009)—it is surprising that no work has been done to identify when shortcut use reaches a frequency denoting fluency with the concept. Arguably, a strong understanding of the inversion concept can only be said to be present when the concept is fluently used to solve the majority of inversion problems; anything less denotes an incomplete understanding of a fundamental mathematical concept.

1.2. The associativity strategy

Individuals' understanding of the associativity concept and its application to problem solving (i.e., the associativity strategy) has received relatively less attention than the inversion concept (Robinson et al., 2006) and no work has been done to identify when its use reaches adult levels. In Grades 1 and 2, 41% of children used their knowledge of associativity to solve 20% of two-term addition problems (Canobi et al., 1998). For Grades 2 through 4, children used the associativity strategy to solve approximately 20% of three-term addition and subtraction associativity problems (Robinson & Dubé, 2009a). Across Grades 6 through 8, children used the associativity strategy on approximately 17% of addition and subtraction associativity problems (Robinson et al., 2006) and on 10% of multiplication and division associativity problems (Robinson & LeFevre, 2012). By adulthood, the associativity strategy is used on approximately 58% of addition and subtraction associativity problems (Robinson & Ninowski, 2003) and on 44% of multiplication and division associativity problems (Dubé & Robinson, 2010a; Robinson & Ninowski, 2003). The results of these studies suggest that individuals' understanding of the associativity concept is not well developed in late childhood; that children's understanding of the associativity concept is relatively weaker than their understanding of the inversion concept; and that children's understanding of the associativity concept is weaker for multiplicative and division than addition and subtraction. Researchers have not yet identified when individuals' understanding of the associativity concept reaches adult levels, a level indicating that the individual can apply the concept fluently and fully understands the associative relationship between operations.

1.3. Individual variability in conceptually-based strategy use

There is individual variability in inversion shortcut and associativity strategy use and this variability changes how we fundamentally understand the development of these two mathematical concepts. Using cluster analysis, studies have identified groups of individuals based on the strategies used to solve the majority (i.e., > 70%) of inversion and associativity problems. In a study of Grade 2, 3, and 4 children's strategy use on three-term addition and subtraction problems, children were classified as users of both the inversion shortcut and associativity strategy (Dual Concept users), users of only the inversion shortcut (Inversion Concept users), or users of neither of the conceptually-based strategies (i.e., using the left-to-right computational strategy $3 + 24 - 22$: $3 + 24 = 27$, $27 - 22 = 5$; No Concept users; Robinson & Dubé, 2009a). In a study of Grade 6, 7, and 8 children's strategy use on multiplication and division inversion problems, children were classified as users of the inversion shortcut, of the negation strategy, or only of the left-to-right computational strategies (Robinson & Dubé, 2009b). In a study of adults' strategy use on multiplication and division problems, participants were classified similarly as in Robinson and Dubé (2009a) with adults classified as Dual, Inversion, or No Concept users (Dubé & Robinson, 2010a). An assessment of all three studies suggests that the pattern of individual variability is somewhat U-shaped; the strategy use of the intermediate group (i.e., children in Grades 6, 7, and 8) demonstrates a weaker understanding of both inversion and associativity than the strategy use of the younger children and adults. Subsets of the younger children and older adults use both conceptually-based strategies to solve the majority of problems whereas no subset of older children do so.

This variability in conceptually-based strategy also suggests that some individuals have a better grasp of the inversion and associativity concepts than their peers. Most research investigating why there is individual variability in inversion shortcut use has focused on the three types of mathematical knowledge, factual, procedural, and conceptual (e.g., Baroody & Lai, 2007; Canobi & Bethune, 2008; Gilmore & Spelke, 2008). The simplest explanation would be that individuals who use conceptually-based strategies have greater factual or procedural mathematical knowledge than individuals who do not use the conceptually-based strategies. However, measures of fluency and procedural skill do not predict inversion shortcut or associativity strategy use (Gilmore & Papadatou-Pastou, 2009; Robinson & Dubé, 2009a,b).

Alternatively, researchers have investigated whether non-mathematic specific abilities play a part in developing children's understanding of the inversion concept. Rasmussen et al. (2003) determined that Grade 1 children's visual-spatial working memory capacity predicted preschool children's inversion shortcut use on concrete inversion problems because it helped the children mentally represent and manipulate the physical blocks. Dubé and Robinson (2010b) measured Grade 6 and 8 children's short-term memory capacity, working memory capacity, and their inversion shortcut use on multiplication and division problems. They found that working memory capacity predicted inversion shortcut use and proposed that greater central executive functioning may have helped the children inhibit alternative, well-practiced but less efficient computational strategies. Whether the use of conceptually-based strategies requires the inhibition of computational strategies had not been empirically investigated at that time but it is theoretically supported.

2. Interruption of procedures mechanism

Siegler and Araya's (2005) strategy choice and discovery simulation model (SCADS*) was proposed to explain the development of inversion shortcut use on three-term addition and subtraction problems and it contains a theoretical framework for how conceptually-based strategies interrupt computational strategies.

According to SCADS*, conceptually-based strategies compete against left-to-right computational strategies and the strategy that is faster,