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Multi-criteria formulations with uncertain satisfactions

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ABSTRACT

We discuss the role that the Choquet integral plays in the aggregation of criteria satisfaction in multi-criteria decision functions. We show how the choice of the associated measure allows for the formulation of many types of multi-criteria decision functions. We note that the need for an ordering of the criteria satisfactions causes difficulties in situations in which there exists a probabilistic type of uncertainty in the knowledge of the criteria satisfactions. We discuss an approach, called the probabilistic exceedance method, for allowing the aggregation of probabilistically satisfied criteria.

1. Introduction

Multi-criteria decision functions arise in many applications (Kahraman, 2008; Greco et al., 2010; Koksalan et al., 2011; Mateo, 2012; Greco et al., 2016). They involve the aggregation of an alternative's individual criteria satisfactions. One approach for building multi-criteria decision functions is with the aid of the Choquet integral (Murofushi and Sugeno, 1993; Mesiar, 1995; Modave and Grabisch; Yager, 1999; Marichal, 2000; Yager, 2004; Grabisch and Labreuche, 2010). One benefit of this approach is that the use of a measure to guide the Choquet approach allows us to model many different types of relationships between the criteria that in turn allows for the modeling of various types of decision imperatives. One source of potential difficulty with the use of the Choquet integral is the need to order the individual criteria satisfaction. While this need for ordering possess no problem when the criteria satisfactions are scalar values the need for ordering can cause some difficulty when the criteria satisfaction are more complex objects then simple scalars. A common example of non-scalar criteria satisfaction is the case where there is some uncertainty about the criteria satisfaction expressed via a probability distribution. Here we face the need to order probability distributions on the unit interval with respect to which is bigger. Specifically it is not always possible to order these probability distributions. Here we investigate a surrogate for the Choquet integral aggregation of probability distributions, called the probabilistic exceedance method, which does not require that we order the probability distributions.

The paper is organized as follows. We first discuss the role the Choquet integral can play in the aggregation of criteria satisfaction in multi-criteria decision functions. We next describe how the choice of measure allows for the formulation of many types of multi-criteria decision functions. We observe that the need for an ordering of the criteria satisfactions in the Choquet integral causes difficulties in situations in which there exists a probabilistic type of uncertainty in the knowledge of the criteria satisfactions. We provide an approach, called the probabilistic exceedance method, for allowing the aggregation of probabilistically satisfied criteria.

2. Aggregating criteria satisfactions using the Choquet integral

Here $\mathbf{C} = \{C_1, \ldots, C_q\}$ are a collection of criteria of interest in a decision. X is a set of alternatives from among which we must select one that best satisfies our criteria. Here for any $x \in X$, $C_i(x) \in [0, 1]$ indicates the degree to which x satisfies the alternative C_i. A common decision procedure is to use a decision function $D(x) = M(C_1(x), C_2(x), \ldots, C_q(x))$ to evaluate each alternative and select the alternative with the largest value for D.

One method for the formulation of the decision function is based on using a fuzzy measure to express the importance relationship between the criteria. Here we take a weighted average of the criteria satisfactions using the Choquet integral (Beliakov et al., 2007).

A fuzzy measure (Sugeno, 1977; Wang and Klir, 1992; Grabisch et al., 2000; Wang and Klir, 2009) on the space of criteria, $\mathbf{C}$ is a mapping μ: $2^{\mathbf{C}} \to [0, 1]$ having the properties: 1. $\mu(\varnothing) = 0$, 2. $\mu(\mathbf{C}) = 1$ and 3. if $A \supset B$ then $\mu(A) \geq \mu(B)$. We see a fuzzy measure μ associates with subsets of $\mathbf{C}$ a value from the unit interval that is monotonic in the sense

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that a smaller set cannot have a bigger value than a larger set. In the following we shall follow the policy of simply using the term measure with the understanding that we are referring to a fuzzy measure.

In the framework of multi-criteria decision making if A is a subset of criteria the term $\mu(A)$ indicates the importance associated with the subset A of criteria. From conditions 1 and 2 we see that the whole set of criteria, $\mathbf{C}$, has collective importance of one while the empty set has importance of zero. Furthermore from condition 3 we see a smaller collection of criteria cannot have more importance than a larger set.

Here we use the Choquet integral in coordination with a fuzzy measure, which captures the importance relationship between the criteria, to construct a decision function $D(x) = M(C_1(x), C_2(x), \ldots, C_q(x))$ (Beliakov et al., 2007). In particular

$$D(x) = \mathrm{Choq}_\mu(C_1(x), \ldots, C_q(x)) = \sum_{j=1}^{q}(\mu(H_j) - \mu(H_{j-1}))C_{\rho(j)}(x)$$

where $\rho(j)$ is an index function so that $C_{\rho(j)}$ is the criteria with the jth largest satisfaction by x. Thus $C_{\rho(1)}(x) \geq C_{\rho(2)} \geq \ldots, \geq C_{\rho(q)}(x)$. Furthermore $H_j = \{C_{\rho(k)}/k = 1 \text{ to } j\}$, it is the subset of the criteria with the j largest satisfactions to x. Central to the use of the Choquet integral is an ordering of the criteria by their satisfaction by x.

We observe that the H_j forms a chain, that is $\emptyset \subseteq H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_q = \mathbf{C}$ and that $\mathrm{Card}(H_j) = j$. We note that since $H_0 = \emptyset$ and $H_q = \mathbf{C}$ we have $\mu(H_0) = 0$ and $\mu(H_q) = 1$. Based on the monotonicity of the measure μ we have the $\mu(H_j) \geq \mu(H_{j-1})$ for $j = 1$ to q. Here then we see that $0 \leq \mu(H_j) - \mu(H_{j-1}) \leq 1$. With $w_j = \mu(H_j) - \mu(H_{j-1})$ we can express

$$D(x) = \sum_{j=1}^{q} w_j\, C_{\rho(j)}(x).$$

Further we see that $\sum_{j=1}^{q} w_j = \sum_{j=1}^{q}(\mu(H_j) - \mu(H_{j-1})) = \mu(H_q) - \mu(H_0) = \mu(\mathbf{C}) - \mu(\emptyset) = 1$. Thus here we see that the w_j are collection of weights lying in the unit interval and summing to one. Thus we can conclude that the Choquet integral, $D(x) = \sum_{j=1}^{q} w_j\, C_{\rho(j)}(x)$, provides a weighted average of the criteria satisfaction. A unique feature of this weighted average is that the weights are dependent on the measure μ. At times we shall use the notation $D_\mu(x)$ to emphasize that weights are based on the measure μ.

It can be shown that the decision function $D_\mu(x)$ based on the use of the Choquet integral is an aggregation operator (Beliakov et al., 2007). That is for any measure μ

(1) $D_\mu(x) = \mathrm{Choq}_\mu(0, 0, \ldots, 0) = 0$
(2) $D_\mu(x) = \mathrm{Choq}_\mu(1, \ldots, 1) = 1$
(3) $D_\mu(x) = \mathrm{Choq}_\mu(C_1(x), \ldots, C_q(x)) \geq D_\mu(y) = \mathrm{Choq}_\mu(C_1(y), \ldots, C_q(y))$ if for all C_i we have that $C_i(x) \geq C_i(y)$.

If alternative x satisfies all the criteria at least as well as y then its overall satisfaction $D_\mu(x)$ is least as much as that if $D_\mu(y)$.

Furthermore the decision function using the Choquet integral is a mean type aggregation operator (Bullen, 2003), that is $\mathrm{Min}_i[C_i(x)] \leq D_\mu(x) \leq \mathrm{Max}_i[C_i(x)]$. In addition $D_\mu(x)$ is idempotent. Hence if $C_i(x) = a$ for all C_i then $D_\mu(x) = a$.

Thus we see that the decision function based on the use of the Choquet integral to aggregate the criteria satisfactions has many desirable properties including monotonicity.

Using a little bit of algebra we see that

$$D_\mu(x) = \sum_{j=1}^{q}(\mu(H_j) - \mu(H_{j-1}))C_{\rho(j)}(x)D_\mu(x)$$
$$= \sum_{j=1}^{q}(C_{\rho(j)}(x) - C_{\rho(j+1)}(x))\mu(H_j).$$

If μ_1 and μ_2 are two measures on the space C with $\mu_1(A) \geq \mu_2(A)$ for all A we denote this as $\mu_1 \geq \mu_2$ and say μ_1 bigger then μ_2. If $\mu_1 \geq \mu_2$ we can say that μ_1 is more optimistic or generous then μ_2. Using the formulation $D_\mu(x) = \sum_{j=1}^{q}(C_{\rho(j)}(x) - C_{\rho(j+1)}(x))\mu(H_j)$ we see that if $\mu_1 \geq \mu_2$ then $D_{\mu_1}(x) \geq D_{\mu_2}(x)$ for any x.

3. Representing multi-criteria requirements using measures

With the aid of the measure μ on the collection $\mathbf{C} = \{C_1, \ldots, C_q\}$ of relevant criteria we can capture various different types of importance relationships between the criteria.

The most basic case is the additive measure. Here each criteria C_i has an associated value $\alpha_i \in [0, 1]$ corresponding to its importance. With an additive measure for any subset A of $\mathbf{C}$ we have $\mu(A) = \sum_{i, C_i \in A} \alpha_i$. We see that $\mu(\{C_i\}) = \alpha_i$ and since we require that $\mu(\mathbf{C}) = 1$ then we have $\sum_{i=1}^{q} \alpha_i = 1$.

With $\rho(j)$ being an index function so that $C_{\rho(j)}(x)$ is the jth largest satisfaction by x then $D(x) = \sum_{j=1}^{q}(\mu(H_j) - \mu(H_{j-1}))C_{\rho(j)}(x)$ where $H_j = \{C_{\rho(k)}/k = 1 \text{ to } j\}$. For an additive measure μ since $\mu(H_j) = \sum_{k=1}^{j} \alpha_{\rho(k)}$ and $\mu(H_{j-1}) = \sum_{k=1}^{j-1} \alpha_{\rho(k)}$ we have $\mu(H_j) - \mu(H_{j-1}) = \alpha_{\rho(j)}$ and hence

$$D_\mu(x) = \sum_{j=1}^{q} \alpha_{\rho(j)} C_{\rho(j)}(x) = \sum_{i=1}^{q} \alpha_i C_i(x).$$

It is the simple weighted average of the criteria satisfactions.

One special case of additive measure is one where $\alpha_i = 1/q$ for all C_i. It this case all criteria are of equal importance. Another special case of additive measure is one in which $\alpha_K = 1$ and $\alpha_i = 0$ for $i \neq K$. Here criteria C_K has complete importance, no other criteria has any importance. An additive measure with $\alpha_K = 1$ and $\alpha_i = 0$ for $i \neq K$ can be seen as a measure focused on C_K. Any subset of criteria A containing C_K has collective importance $\mu(A) = 1$ while any subset A not containing C_K has $\mu(A) = 0$. We easily that for this measure with $\alpha_K = 1$ that $D(x) = \sum_{i=1}^{q} \alpha_i C_i(x) = C_K(x)$. Thus the degree of satisfaction is simply the degree of satisfaction of x to criteria C_K.

Fundamental to the preceding was the assumption that the α_i in addition to lying in the unit interval they summed to one, $\sum_{i=1}^{q} \alpha_i = 1$. Here we shall consider the situation where $\sum_{i=1}^{q} \alpha_i \neq 1$, the sum of the α_i is not one. We shall look at two approaches to address this problem.

One approach is a kind of normalization. Here with $T = \sum_{i=1}^{q} \alpha_i$ we replace each α_i with $\beta_i = \frac{\alpha_i}{T}$. In this case $\sum_{i=1}^{q} \beta_i = 1$ and we are back to a basic additive measure, $\widetilde{\mu}$ where $\widetilde{\mu}(A) = \sum_{i \in A} \beta_i$. In this case we have $D(x) = \sum_{i=1}^{q} \beta_i\, C_i(x) = \frac{1}{T}\sum_{i=1}^{q} \alpha_i C_i(x)$.

Another way of handling situation where we have the individual criteria importances that do not sum to one is to use a non-strictly additive measure which does not require the sum of the weights to be one such as the Sugeno measure (Sugeno, 1977; Klir, 2006), a measure closely related to the additive measure. The Sugeno measure is defined to satisfy, for all A and $B \subseteq \mathbf{C}$ with $A \cap B = \emptyset$, the equation

$$\mu_\lambda(A \cup B) = \mu_\lambda(A) + \mu_\lambda(B) + \lambda\mu_\lambda(A)\mu_\lambda(B) \text{ for } \lambda > -1.$$

If we denote $\mu_\lambda(\{C_i\}) = g_i \in [0, 1)$ then it can be shown (Sugeno, 1977) that the λ associated with a Sugeno measure can be obtained from equation $(1 + \lambda) = \prod_{i=1}^{q}(1 + \lambda g_i)$. This comes from the fact that $\mu_\lambda(\mathbf{C}) = 1$. Furthermore, it can be shown (Sugeno, 1977) that

$$\text{if } \sum_{i=1}^{q} g_i < 1 \text{ then } \lambda > 1$$
$$\text{if } \sum_{i=1}^{q} g_i = 1 \text{ then } \lambda = 0$$
$$\text{if } \sum_{i=1}^{q} g_i > 1 \text{ then } \lambda \in (-1, 0).$$

We note that if $\sum_{i=1}^{q} g_i = 1$ then this becomes the usual additive measure since $\lambda = 0$.

Using the Sugeno measure we obtain a variation of the additive measure for the case where all $\alpha_i \in [0, 1)$ and $\sum_{i=1}^{q} \alpha_i \neq 0$. Here we let $g_i = \alpha_i$ and obtain a Sugeno measure with λ obtained by solving the equation $(1 + \lambda) = \prod_{i=1}^{q}(1 + \lambda\alpha_i)$.

Let us look at the associated form of the decision function $D_{\mu_\lambda}(x)$ in this case where we let $g_i = \alpha_i$. Here we note that with $H_j = \{C_{\rho(k)}/k$