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Aggregation in the analytic hierarchy process: Why weighted geometric mean should be used instead of weighted arithmetic mean

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ABSTRACT

The main focus of this paper is the aggregation of local priorities into global priorities in the Analytic Hierarchy Process (AHP) method. We study two most frequently used aggregation approaches - the weighted arithmetic and weighted geometric means - and identify their strengths and weaknesses. We investigate the focus of the aggregation, the assumptions made on the way, and the effect of different normalizations of local priorities on the resulting global priorities and their ratios. We clearly show the superiority of the weighted geometric mean aggregation over the weighted arithmetic mean aggregation in AHP for the purpose of deriving global priorities of alternatives. We also contribute to the literature on rank reversal in AHP. In particular, we show that a change of the normalization condition for the local priorities of alternatives may result in different ranking when the weighted arithmetic mean aggregation is used for deriving global priorities of alternatives, and we demonstrate that the ranking obtained by the weighted geometric mean aggregation is not normalization dependent. Moreover, we prove that the ratios of global priorities of alternatives obtained by the weighted geometric mean aggregation are invariant under the normalization of local priorities of alternatives and weights of criteria. We also propose three alternative approaches to aggregating preference information contained in local pairwise comparison matrices of alternatives into a global consistent pairwise comparison matrix of alternatives and prove their equivalence.

1. Introduction

Analytic Hierarchy Process (AHP) is a well-known multi-criteria decision-making method developed by Saaty (1977, 1980). The body of literature and research on AHP is extensive. AHP is continuously being used to support decisions in important decision problems in various fields ranging from engineering and industry applications (Duman, Tozanli, Kongar, & Gupta, 2017), through social sciences applications (Jandová, Krejčí, Stoklasa, & Fedrizzi, 2017; Saaty, 2013) to applications in the medical sector (Nazari, Fallah, Kazemipoor, & Salehipour, 2018); for a more comprehensive overview see, e.g., the literature reviews by de F. S. M. Russo and Camanho (2015); Liberatore and Nydick (2008); Subramanian and Ramanathan (2012); Vaidya and Kumar (2006). Combinations of AHP with other methods in practical decision-making are also frequent; see, e.g., the literature reviews by Ho (2008); Ho and Ma (2018). Thus, any flaw in the AHP methodology may lead to incorrect decisions with tremendous negative consequences.

AHP is based on structuring the decision problem into a problem hierarchy and pairwisely comparing objects in one level of the hierarchy with respect to the superior object from the upper level of the hierarchy. Based on the pairwise comparisons of objects, local priorities of objects are derived and aggregated within the hierarchy in order to derive global priorities. For the simplicity of explanation and without any loss of generality, we will assume a multi-criteria decision-making problem with a simple 3-level hierarchy, i.e. a problem with the goal specified in the first level of the hierarchy, n criteria $C_1, \ldots, C_n$ relevant to the problem specified in the second level of the hierarchy, and m alternatives $A_1, \ldots, A_m$ specified in the third level of the hierarchy. The generalization of our findings to hierarchies with more levels is straightforward.

A diagram describing the main stages of AHP is provided in Fig. 1. Note that the last stage, i.e. the construction of a global PCM of alternatives, is usually not done and the global priorities of alternatives are used directly to rank the alternatives. In this paper,

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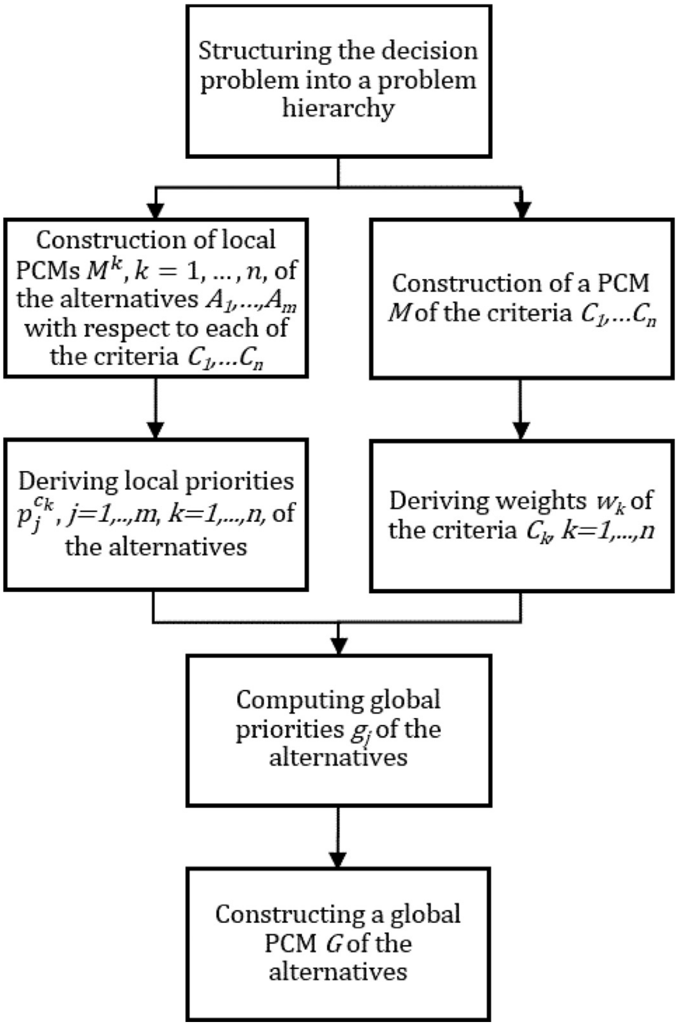

Fig. 1. Diagram of AHP stages.

however, we will consider also this optional last step that is important for the correct interpretation of the final priorities of alternatives.

The main idea of AHP as introduced by Saaty (1980) is that instead of providing the priorities w_i and w_j of objects o_i and o_j, respectively, a single number representing the ratio w_i/w_j is provided by the decision maker. Thus, the decision maker provides a pairwise comparison m_{ij}, $i, j \in \{1, \cdots, n\}$, of objects o_i and o_j according to their relative priorities belonging to a ratio scale that is supposed to approximate the ratio w_i/w_j, i.e., $m_{ij} \approx w_i/w_j$. In real-world applications, pairwise comparisons are usually done by choosing linguistic terms from predefined Saaty's scale (Saaty, 1980). The pairwise comparisons of objects in one level of the hierarchy with respect to the superior object from the upper level of the hierarchy are usually structured into a pairwise comparison matrix (PCM).

Definition 1. A PCM of n objects is a square matrix $M = \{m_{ij}\}_{i,j=1}^n$ whose element m_{ij} expresses the relative preference (or importance) of object o_i over object o_j, and as such approximates the ratio w_i/w_j, i.e., $m_{ij} \approx w_i/w_j$, where $w_1, \ldots, w_n$ are the possibly unknown priorities of the objects.

For particular examples of PCMs, see, e.g., Table 1.

Table 1
Local consistent PCMs of alternatives with respect to criteria C_1 and C_2.

PCM M^1 of alternatives w.r.t. criterion C_1				PCM M^2 of alternatives w.r.t. criterion C_2			
	A_1	A_2	A_3		A_1	A_2	A_3
A_1	1	2	8	A_1	1	1/3	1/3
A_2	1/2	1	4	A_2	3	1	1
A_3	1/8	1/4	1	A_3	3	1	1

Both PCMs M^1 and M^2 are fully consistent according to (1). The non-normalized local-priorities vectors computed for the alternatives with respect to criteria C_1 and C_2 by the GMM are $\underline{p}^{C_1} = (2.520, 1.260, 0.315)$ and $\underline{p}^{C_2} = (0.481, 1.442, 1.442)$, respectively.

Definition 2. (Saaty, 1980) A PCM $M = \{m_{ij}\}_{i,j=1}^n$ is said to be consistent if it satisfies the multiplicative-transitivity property

$$m_{ij} = m_{ik} \cdot m_{kj}, \qquad i, j, k = 1, \ldots, n. \tag{1}$$

For particular examples of consistent PCMs, see, e.g., Table 1.

Proposition 1. *(Saaty, 1994) A PCM $M = \{m_{ij}\}_{i,j=1}^n$ is consistent if and only if there exists a positive vector $\underline{w} = (w_1, \ldots, w_n)^T$ such that*

$$m_{ij} = w_i/w_j, \qquad i, j = 1, \ldots, n. \tag{2}$$

Notice that from Proposition 1 the reciprocity property $m_{ij} = 1/m_{ji}$, $i, j = 1, \cdots, n$, of consistent PCMs follows; $m_{ij} = \frac{w_i}{w_j} = \frac{1}{\frac{w_j}{w_i}} = \frac{1}{m_{ji}}$. The reciprocity property is required from inconsistent PCMs as well. Thus, in practice, the decision maker provides only one of the PCs m_{ij} and m_{ji}, $i, j \in \{1, \ldots, n\}$, and the other one is then inserted automatically into the PCM by using the reciprocity property $m_{ij} = 1/m_{ji}$.

The priorities $w_1, \ldots, w_n$ of objects are usually derived from a PCM by the maximal eigenvector method (EVM) or by the geometric mean method (GMM). In both cases, the priorities $w_1, \ldots, w_n$ are such that they satisfy (2) in case of a consistent PCM M or $m_{ij} \approx w_i/w_j$, $i, j = 1, \cdots, n$, in case of an inconsistent PCM M. The EVM was introduced by Saaty (1980) in the original version of AHP. According to the EVM, the priorities of objects are derived as the components of the maximal eigenvector $\underline{w} = (w_1, \ldots, w_n)^T$ of the PCM M, i.e. as the solutions to the equation

$$M\underline{w} = \lambda \underline{w}, \tag{3}$$

where λ is the maximal eigenvalue of the PCM M. According to the GMM, the priorities of objects are derived as the geometric means of the pairwise comparisons in the rows of the PCM M (Barzilai, d. Cook, & Golany, 1987), i.e., as

$$w_i = \sqrt[n]{\prod_{j=1}^{n} m_{ij}}, \qquad i = 1, \ldots, n. \tag{4}$$

Note 1. The priority vector $\underline{w} = (w_1, \ldots, w_n)^T$ computed by (4) is in fact (up to the multiplication by a scalar) a solution to the problem of finding the minimum of the function

$$f(w_1, \ldots, w_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\ln(m_{ij}) - \ln\left(w_i/w_j\right)\right)^2$$

in the logarithmic least squares method introduced by Crawford and Williams (1985). The approach to the computation of priorities of objects by the GMM can therefore be understood as finding priorities of objects such that their ratios are as close as possible to the respective elements of the PCM M.