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A note on initial value problems for fractional fuzzy differential equations

Ngo Van Hoa [a,b], Vasile Lupulescu [c], Donal O'Regan [d]

[a] *Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Viet Nam*

[b] *Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Viet Nam*

[c] *Constantin Brancusi University, Republicii 1, 210152 Targu-Jiu, Romania*

[d] *School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland*

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Abstract

In this note, we present some remarks on solutions of fractional fuzzy differential equation. In general, a fractional fuzzy differential equation and a fractional fuzzy integral equation are not equivalent. In this paper we give an appropriate condition so that this equivalence is valid. Some examples are provided to illustrate the theory.

Keywords: Fractional calculus; Fractional fuzzy differential equations; Fractional fuzzy integral equations

1. Introduction

Fuzzy analysis and fuzzy differential equations were proposed to handle uncertainty due to incomplete information that appears in many mathematical or computer models of some deterministic real-world phenomena. Recently this theory has been further developed and a wide number of applications of this theory have been considered. As a result fuzzy fractional calculus and fuzzy fractional differential equations has emerged as the significant topic (see [7–9]). The concept of fuzzy type Riemann–Liouville differentiability based on Hukuhara differentiability was initiated in [1,2] and using the Hausdorff measure of non-compactness the authors established the existence to some fuzzy integral equations using appropriate compactness type conditions. Various approaches and methods, based on Hukuhara differentiability or generalized Hukuhara differentiability (see [8]), were then considered in a number of papers in the literature (see for example [3,4,12,14–17,20,21]) and we now summarize briefly some of these results. The authors in [6] established some results on the existence and uniqueness of solutions to fuzzy fractional differential equations under Hukuhara fractional Riemann–Liouville differentiability and the authors in [4,5] discussed the gen-

E-mail address: ngovanhoa@tdt.edu.vn (N.V. Hoa).

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eralized Hukuhara fractional Riemann–Liouville and Caputo differentiability of fuzzy-valued functions. In [17] the authors considered the solution to fuzzy fractional initial value problem under Caputo generalized Hukuhara differentiability using a modified fractional Euler method and the authors in [18] established the existence and uniqueness of a solution to fuzzy fractional differential equation under a Caputo type-2 fuzzy fractional derivative and in particular presented a definition of the Laplace transform of type-2 fuzzy number-valued functions. In [19] the author presented mathematical foundations for studies of random fuzzy fractional integral equations which involve a fuzzy integral of fractional order and in [20] the authors established the existence and uniqueness of solutions to a fuzzy fractional initial value problem under Caputo generalized Hukuhara differentiability and in particular they discussed numerical approximation of solutions using the product trapezoidal and product rectangle formulas.

In the theory of fuzzy differential equations using the Hukuhara generalized difference two forms of the associated fuzzy integral equation to the initial value problem is considered (see [8]). The first one leads to the solutions having nondecreasing diameter of their values (d-increasing), and the second leads to solutions have nonincreasing diameter of the values (d-decreasing). Consider the initial value problem

$$x'(t) = f(t, x(t)), \qquad x(a) = x_0 \in E, \tag{1}$$

where $f : [a, b] \times E \to E$ is a continuous fuzzy function, E is the space of fuzzy numbers and x' denotes the generalized derivative of the fuzzy-valued function x. It is well-known that the fuzzy differential equation (1) is equivalent to the fuzzy integral equation

$$x(t) \ominus_{gH} x_0 = \int_a^t f(s, x(s))ds,\ t \in [a, b], \tag{2}$$

where $\ominus_{gH}$ denotes the generalized Hukuhara difference.

If for every $r \in [0, 1]$ the function $t \mapsto d\left([x(t)]^r\right)$ is nondecreasing for all $t \in [a, b]$, then (2) can be rewritten as

$$x(t) = x_0 + \int_a^t f(s, x(s))ds,\ t \in [a, b].$$

Since $t \mapsto \int_a^t f(s, x(s))ds$ is d-increasing on $[a, b]$, then it follows that $t \mapsto x(t)$ is also d-increasing on $[a, b]$.

If for every $r \in [0, 1]$ the function $t \mapsto d\left([x(t)]^r\right)$ is nonincreasing for all $t \in [a, b]$, then (2) can be rewritten as

$$x_0 \ominus x(t) = \int_a^t (-f(s, x(s)))ds,\ t \in [a, b],$$

where $\ominus$ denotes the Hukuhara difference. Since $t \mapsto \int_a^t (-f(s, x(s)))ds$ is also d-increasing on $[a, b]$, then from Lemma 2 in our paper it follows that $t \mapsto x(t)$ is d-decreasing on $[a, b]$. Furthermore, it is clear that, a unique solution of (2) (if it exists) is also a unique solution of (1) in each case.

However, in the theory of fuzzy fractional differential equations there is a minor drawback in some papers as the assertion of the equivalence between fractional fuzzy differential equations and fractional fuzzy integral equations is not correct and we note that in general a solution of a fractional fuzzy integral equation is not a solution of a fractional fuzzy differential equation. In this note we present some remarks on solutions of fractional fuzzy differential equation under Caputo generalized Hukuhara differentiability. In particular, we show that in general a fractional fuzzy differential equation and a fractional fuzzy integral equation are not equivalent and we give an appropriate condition when this equivalence is valid. Some examples are given to illustrate the theory.

The remainder of this note is organized as follows. Section 2 reviews basic concepts and present notation used in this text. Section 3 presents some remarks on solutions of fractional fuzzy differential equations. Section 4 presents some concluding remarks.

2. Fundamental theorems of fuzzy fractional analysis

Let E be the class of fuzzy numbers, i.e. normal, convex, upper semicontinuous and compactly supported fuzzy subsets of the real numbers. For $r \in (0, 1]$, denote $[\omega]^r = \{z \in \mathbb{R} \mid \omega(z) \geq r\}$ and $[\omega]^0 = \overline{\{z \in \mathbb{R} \mid \omega(z) > 0\}}$. Then it is