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Some notes on the partial orders induced by a uninorm and a nullnorm in a bounded lattice

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Received 14 April 2017; received in revised form 23 August 2017; accepted 23 August 2017

Abstract

In this paper, an equivalence on the class of nullnorms on a bounded lattice is discussed. Some relationships between the U and V-partial orders are investigated. A necessary and sufficient condition making a bounded lattice also a lattice w.r.t. the V-partial order is given. Moreover, the relation between the algebraic structures related to the U and V-partial orders is studied.

Keywords: Nullnorm; Bounded lattice; Partial order; Equivalence; Uninorm; Nullnorm

1. Introduction

Nullnorms [6,7,9,25] and uninorms [8,27,29] with more applications like fuzzy logic, expert systems, neural networks and fuzzy system modelling [10,13,31,26] are the important aggregation operators generalizing the notions of t-norms and t-conorms [11,30].

In recent years, inducing an order from logical operators has been an interesting problem for many researchers [1–3,12,15,18–22]. In this sense, in [18], T-partial order, induced by t-norms, has been defined on a bounded lattice L as follows: For any elements $x, y \in L$

$$x \preceq_T y \Leftrightarrow T(\ell, y) = x, \text{ for some } \ell \in L.$$

In [12], S-partial order, induced by t-conorms, has been given in a similar way, that is, for any $x, y \in L$

$$x \preceq_S y \Leftrightarrow S(\ell^*, x) = y, \text{ for some } \ell^* \in L.$$

Later on, in [1,12], V and U-partial orders, respectively induced by nullnorms and uninorms, have been introduced and some basic properties have been investigated.

In [2,3], an equivalence relation on the class of t-norms on a bounded lattice based on the equality of the sets admitting some incomparability w.r.t. the T-partial order has been defined. In [1], the similar equivalence relation

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http://dx.doi.org/10.1016/j.fss.2017.08.010
0165-0114/© 2017 Published by Elsevier B.V.

has been investigated for the V-partial order on the unit interval [0, 1]. Also, in [1], it has been shown that the U and V-partial orders do not coincide even if they have the same underlying t-norms and t-conorms. Nevertheless, the conditions under which the U and V-partial orders coincide have not been investigated.

In this paper, we investigate an equivalence on the class of nullnorms on a bounded lattice based on the equality of the sets, denoted by K_V, admitting some incomparability w.r.t. the V-partial order and we present the relations between the equivalence classes of nullnorms and the equivalence classes of their underlying t-norms and t-conorms. Moreover, we determine some conditions under which the U and V-partial orders coincide. The paper is organized as follows:

We shortly recall some basic notions and results in Section 2.

In Section 3, we give some relationships between the U and V-partial orders. We present a necessary and sufficient condition making a bounded lattice L also a lattice w.r.t. the V-partial order. We determine a relation between the algebraic structures related to the U and V-partial orders.

In Section 4, we define an equivalence on the class of nullnorms on a bounded lattice with a zero element a based on the equality of the sets admitting some incomparability w.r.t. the V-partial order and we determine that two idempotent nullnorms are equivalent. We prove that two nullnorms are equivalent when their underlying t-norms and t-conorms are equivalent. We give an example illustrating that the underlying t-norms and t-conorms need not be equivalent even if two nullnorms are equivalent. Also, we show that for the equivalence of two nullnorms a necessary and sufficient condition is the equivalence of their underlying t-norms and t-conorms on a bounded lattice all elements comparable with a. We present a relationship between the sets admitting some incomparability w.r.t. the T-partial order, induced by a t-norm T, and the S-partial order, induced by the N-dual t-conorm of T. Also, we obtain that for the equivalence of any two t-norms, a necessary and sufficient condition is the equivalence of their N-dual t-conorms. We determine the equivalence classes of the greatest and smallest nullnorms on the unit interval [0, 1]. We give a relationship between the sets K_V and K_{V_ϕ}, where V_ϕ is the ϕ-conjugate of V. Finally, we characterize the set K_V under some conditions.

2. Notations, definitions and a review of previous results

In this section, we recall some basic notions and results.

Definition 1. [18,24] An operation T (S) on a bounded lattice L is called a triangular norm (triangular conorm) if it is commutative, associative, increasing w.r.t. the both variables and has a neutral element 1 (0).

Definition 2. [18,21] A t-norm T (or a t-conorm S) on a bounded lattice L is divisible if the following condition holds:
For all $x, y \in L$ with $x \leq y$ there is $z \in L$ such that $x = T(y, z)$ (or $y = S(x, z)$).

Definition 3. [16] Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $V : L^2 \to L$ is called a nullnorm on L, if it is commutative, associative, increasing w.r.t. the both variables and has a zero (absorbing) element $a \in L$ such that for all $x \leq a$, $V(x, 0) = x$ and for all $x \geq a$, $V(x, 1) = x$.

In this study, the notation $\mathcal{V}(a)$ will be used for the set of all nullnorms on L with a zero element $a \in L$.

The greatest and smallest nullnorm on [0, 1] are defined by, respectively,

$$V_a^{(\vee)}(x, y) = \begin{cases} \max(x, y) & (x, y) \in [0, a]^2, \\ a & (x, y) \in [a, 1)^2 \cup D_a, \\ \min(x, y) & \text{otherwise}, \end{cases} \quad \text{and}$$

$$V_a^{(\wedge)}(x, y) = \begin{cases} \min(x, y) & (x, y) \in [a, 1]^2, \\ a & (x, y) \in (0, a]^2 \cup D_a, \\ \max(x, y) & \text{otherwise}, \end{cases}$$

where $D_a = [0, a) \times (a, 1] \cup (a, 1] \times [0, a)$.

Proposition 1. [16] *Let $(L, \leq, 0, 1)$ be a bounded lattice, and $V \in \mathcal{V}(a)$. Then*
(i) $S_V = V|_{[0,a]^2} : [0, a]^2 \to [0, a]$ *is a t-conorm on* $[0, a]$.
(ii) $T_V = V|_{[a,1]^2} : [a, 1]^2 \to [a, 1]$ *is a t-norm on* $[a, 1]$.