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On the properties of the F-partial order and the equivalence of nullnorms

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Received 4 February 2017; received in revised form 31 October 2017; accepted 8 November 2017

Abstract

In this study, we investigate a partial order induced by a nullnorm F on a bounded lattice L, called the F-partial order, which was recently introduced by Aşıcı. We consider the case where $L = [0, 1]$. First, we define and comprehensively study the set of all incomparable elements with arbitrary but fixed $x \in (0, 1)$ according to the F-partial order. We then define and discuss the equivalence relation on the class of nullnorms according to the partial orders that they generate. By defining this relation, the equivalence relations defined on the class of t-norms and t-conorms are extended to a more general form. Finally, we provide an answer to an open problem regarding the relation between the lattice order and the F-partial order on $[0, 1]$.

Keywords: Equivalence; Nullnorm; Partial order

1. Introduction

In 1960, the triangular norms (t-norms) and triangular conorms (t-conorms) were first introduced by Schweizer and Sklar in [26]. Subsequently, they have been used in many applications in the areas of fuzzy set theory and fuzzy logic. The concepts of a nullnorm and t-operator were introduced by [20] and [5], respectively. The equivalence of nullnorms and t-operators was stated by [21].

In [23], a natural order for semigroups was defined. Similarly, in [15], a partial order defined by the means of t-norms on a bounded lattice was introduced. This partial order is called a T-partial order on a bounded lattice.

An order induced by a nullnorm on a bounded lattice was defined by [1]. In addition, the set of incomparable elements with respect to the F-partial order for any nullnorm on $[0, 1]$ was defined, which is called K_F. Nullnorms and t-norms have also been investigated in many studies by [2,3,7–9,12,13,18–22,27].

In the present study, we investigate some properties of an order induced by nullnorms, called the F-partial order. The remainder of this paper is organized as follows. In Section 2, some basic notions are presented in brief. In Section 3, we define the set $\mathcal{I}_F{}^{(x)}$, which denotes the set of all incomparable elements with arbitrary but fixed $x \in (0, 1)$ according to the F-partial order, and we investigate some properties of this set. In Section 4, we define an equivalence

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https://doi.org/10.1016/j.fss.2017.11.008
0165-0114/© 2017 Published by Elsevier B.V.

Please cite this article in press as: E. Aşıcı, On the properties of the F-partial order and the equivalence of nullnorms, Fuzzy Sets Syst. (2017), https://doi.org/10.1016/j.fss.2017.11.008

on the class of nullnorms on the unit interval [0, 1] based on the equality of the their partial orders. We also determine the equivalence classes of the smallest and greatest nullnorms on [0, 1]. Next, we introduce the concept of a conjugate of a nullnorm and we present the sufficient and necessary condition such that a nullnorm and its conjugate nullnorm are equivalent. In [1], the following open problem was posed. Given an $(a, b) \subsetneq (0, 1)$, can we find a nullnorm F on the unit interval [0, 1] such that $K_F = (a, b)$? In Section 5, we provide an answer to this open problem.

2. Preliminaries

To ensure that this study is self-contained, we first recall some concepts and results that are used in the following.

Definition 1. [25] Let $(L, \leq, 0, 1)$ be a bounded lattice. A *triangular norm* T (t-norm) is a binary operation on L that is commutative, associative, monotone, and has neutral element 1.

Example 1. [17] The four basic t-norms T_M, T_P, T_L, and T_D on [0, 1] are given by:

$$T_M(x, y) = \min(x, y),$$
$$T_P(x, y) = xy,$$
$$T_L(x, y) = \max(x + y - 1, 0),$$
$$T_D(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, 1)^2\ , \\ \min(x, y), & \text{otherwise.} \end{cases}$$

Definition 2. [25] Let $(L, \leq, 0, 1)$ be a bounded lattice. A *t-conorm* S is a binary operation on L that is commutative, associative, monotone, and has neutral element 0.

Example 2. [17] The four basic t-conorms S_M, S_P, S_L, and S_D on [0, 1] are given by:

$$S_M(x, y) = \max(x, y),$$
$$S_P(x, y) = x + y - xy,$$
$$S_L(x, y) = \min(x + y, 1),$$
$$S_D(x, y) = \begin{cases} 1 & \text{if } (x, y) \in (0, 1]^2\ , \\ \max(x, y), & \text{otherwise.} \end{cases}$$

The extremal t-norms $T_\wedge$ and T_W on L are defined as follows:

$$T_\wedge(x, y) = x \wedge y$$
$$T_W(x, y) = \begin{cases} x & \text{if } y = 1\ , \\ y & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the t-conorms $S_\vee$ and S_W can be defined as above.

In particular, we have $T_W = T_D$ and $T_\wedge = T_M$ for $L = [0, 1]$.

Example 3. [17] The t-norm T^{nM} on [0, 1] is defined as follows:

$$T^{nM}(x, y) = \begin{cases} 0 & x + y \leq 1\ , \\ \min(x, y) & \text{otherwise.} \end{cases}$$

$T^n M$ is called the nilpotent minimum t-norm.

The t-norm T^* on [0, 1] is defined as follows:

$$T^*(x, y) = \begin{cases} 0 & (x, y) \in (0, k)^2\ , \\ \min(x, y) & \text{otherwise,} \end{cases} \qquad 0 < k < 1.$$

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