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Back-and-forth systems for fuzzy first-order models

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Abstract

This paper continues the study of model theory for fuzzy logics by addressing the fundamental issue of classifying models according to their first-order theory. Three different definitions of elementary equivalence for fuzzy first-order models are introduced and separated by suitable counterexamples. We propose several back-and-forth conditions, based both on classical two-sorted structures and on non-classical structures, that are useful to obtain elementary equivalence in particular cases as we illustrate with several examples.

Keywords: Mathematical fuzzy logic; First-order fuzzy logics; Non-classical logics; Elementary equivalence; Back-and-forth systems; Model theory

1. Introduction

Starting from several seminal works (e.g. [33,31,24]) and firstly systematized in Petr Hájek's landmark monograph [22], mathematical fuzzy logic (MFL) has been developed as a study of logical systems able to handle graded properties (and related notions of partial truth, vagueness, fuzziness, imprecision, etc.). A considerable amount of papers in the area have resulted in a deep knowledge of, mainly, propositional fuzzy logics as collected in the series of handbooks [8]. However, propositional logics may be seen as insufficient for some purposes because they involve only truth-functional connectives and are not expressive enough to model some important problems in computer science and to provide a satisfactory tool for knowledge representation and analysis of reasoning with graded predicates.

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There are indeed important areas in fuzzy set theory that deal with first-order notions and thus can be formalized in first-order predicate fuzzy logics such as fuzzy graphs [29], valued preference modeling [19], or fuzzy orders and similarities [2].

Predicate fuzzy logics in the full first-order language with universal and existential quantifiers and an arbitrary stock of functional and relational symbols were already introduced in Hájek's initial works. However, only recently they have become the object of systematic research. The papers [23,9] give axiomatizations and completeness theorems in a very general framework for graded logics and lay the foundations for an incipient model theory of such logics.

Elementary equivalence is a central notion in classical model theory that allows to classify models by identifying those that validate the same first-order sentences. It was introduced by Tarski [34] and later used by himself and Vaught to study elementary extensions and elementary chains [35]. Inspired by the proof of Cantor's theorem on countable dense linear orderings without endpoints [25,27], back-and-forth systems of partial isomorphisms were introduced to model theory, and a characterization of elementary equivalence in terms of these systems was introduced in Fraïssé [20,21]. Independently, the notion was characterized using games in Ehrenfeucht [18]. For general surveys on the subject and historical overviews we refer the reader to [1,5,17,26].

In the context of first-order fuzzy logics, the notion of elementary equivalence was defined in [23, Definition 10], where it was used to characterize conservative extensions of theories. A series of papers have followed this definition in various contexts: characterization of strong completeness with respect to models based on a particular class of algebras [7], study of mappings and diagrams [11], ultraproduct constructions [12], characterization of elementary equivalence in terms of elementarily mappings [14], characterization of elementarily classes as those closed under elementary equivalence and ultraproducts [13], and Löwenheim–Skolem theorems for non-classical logics [15]. An alternative approach considers models of first-order fuzzy logics with evaluated syntax [32,30]. Finally, an independent, but related, stream of research is that of continuous model theory, in which the underlying logic is essentially Łukasiewicz logic expanded with connectives for each continuous function [6,4].

The goal of this paper is to contribute to the understanding of elementary equivalence of models of first-order fuzzy logics by focusing on two aspects:

a) considering three different possible generalizations of the classical notion of elementary equivalence to the fuzzy case, that were not distinguished in the previous literature, and
b) providing sufficient back-and-forth conditions to prove elementary equivalence of fuzzy models in particular cases.

The paper is organized as follows: after this introduction, Section 2 presents the necessary preliminaries we need recalling several semantical notions from mathematical fuzzy logic, namely, the algebraic counterpart of extensions of the uninorm logic UL, fuzzy first-order models based on such algebras, and some basic model-theoretic notions. Section 3 defines the notions of elementarily equivalent, filter-strong elementarily equivalent, and strongly elementarily equivalent fuzzy models and separates them with natural counterexamples. Section 4 proposes a classical approach to the problem of finding back-and-forth conditions for elementary equivalence of fuzzy models by treating them as classical two-sorted structures. After this, Section 5 presents a genuinely non-classical approach that, based on a syntactical notion of nested rank, allows to build layered back-and-forth systems to prove elementary equivalence of fuzzy models restricted to sentences up to a certain degree of syntactical complexity. Finally, Section 6 ends the paper by discussing a straightforward generalization of the results to a much wider framework and some other concluding remarks.

2. Preliminaries

2.1. Fuzzy first-order models

Model theory studies mathematical structures using different formal languages. In this section we introduce fuzzy first-order models, the object of our study. On the one hand, they provide the semantics of first-order predicate fuzzy logics. On the other hand, fuzzy first-order models can be seen also as two-sorted classical structures. As a general reference for all the notions of mathematical fuzzy logic that we will use in the paper one can consult the handbook [8].

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