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Neighborhood semantics for modal many-valued logics ☆

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Abstract

The majority of works on modal many-valued logics consider Kripke-style possible worlds frames as the principal semantics despite their well-known axiomatizability issues when considering non-Boolean accessibility relations. The present work explores a more general semantical picture, namely a many-valued version of the classical neighborhood semantics. We present it in two levels of generality. First, we work with modal languages containing only the two usual unary modalities, define neighborhood frames over algebras of the logic $\mathrm{FL}_{\mathrm{ew}}$ with operators, and show their relation with the usual Kripke semantics (this is actually the highest level of generality where one can give a straightforward definition of the Kripke-style semantics). Second, we define generalized neighborhood frames for arbitrary modal languages over a given class of algebras for an arbitrary protoalgebraic logic and, assuming certain additional conditions, axiomatize the logic of all such frames (which generalizes the completeness theorem of the classical modal logic E with respect to classical neighborhood frames).

Keywords: Mathematical fuzzy logic; Modal fuzzy logics; Neighborhood frames; Kripke semantics; Many-valued logics

1. Introduction

The study of many-valued propositional logics expanded with modal operators was started by Melvin Fitting in [15,16] and later continued by Petr Hájek and others in the field of Mathematical Fuzzy Logic [19,8] resulting in an active field research (see e.g., [3,4,7,6,5,20,21,23,24,29,30]). In many of these works, since the initial propositional logic may lack an involutive negation, the extended modal system is endowed with two non-interdefinable modal operators, $\Box$ and $\Diamond$, or alternatively one may restrict to the fragment given by only one of these operators. Another

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peculiarity of the syntax of these systems is that, for technical reasons related to the proof of completeness already encountered in Fitting's seminal papers, it often includes truth-constants to denote each element of the intended algebraic semantics. On the other hand, modal fuzzy logics are typically endowed with a relational semantics that generalizes the classical Kripke semantics by allowing a many-valued scale for either (or for both) the truth-values of propositions at each possible world and for the degree of accessibility from one world to another. However, despite its very natural definition, such semantics brings forth serious technical difficulties. Indeed, axiomatizing the Kripke-style semantics over a given algebra (or class of algebras) of truth-values can be in general a complex problem (for instance, no simple axiomatic presentation is known for modal extensions of product logic and one has to resort to the use of truth-constants, $\triangle$ projection, and infinitary rules [30], or for the modal logic over the standard finitary Łukasiewicz logic, which has been axiomatized with an infinitary rule [21]). Conversely, already in the classical case, proof systems with natural syntactic conditions may fail to be complete with any such Kripke-style semantics.

In modal extensions of classical logic, the Scott–Montague neighborhood semantics [25,28] has been used as a more general framework than Kripke frames where, instead of using an accessibility relation, each world is mapped to a set of sets of worlds known as its *neighborhood*. It allows to prove completeness for non-normal modal logics, where the Kripke-style semantics would not work. For analogous reasons, recently some authors have started introducing some notions of neighborhood semantics for modal fuzzy logics. It has been studied in particular settings in [26,27] and in a general framework of fuzzy logics extending MTL (the basic t-norm-based logic [14,22]) in the conference paper [11].

The aim of this paper is to introduce neighborhood semantics for the widest possible class of modal many-valued logics (building on the partial results of [11]) to fulfill the following goals: (1) show the exact relation between the new neighborhood semantics and the usual Kripke-style semantics used so far in modal many-valued logics, (2) assume only the necessary conditions to obtain a semantics that naturally generalizes the classical Scott–Montague semantics and the previous particular proposals for a neighborhood semantics of modal fuzzy logics, and (3) obtain an axiomatization, and the corresponding completeness theorem, of the *global* consequence given by the neighborhood frames defined over an arbitrary class of algebras. Unlike in classical logic, there is no straightforward relationship between the global and the local consequence and, hence, the study of the latter is left for a future investigation.

To achieve the first goal it suffices to formulate our new notions in the usual framework of modal many-valued logics with Kripke frames, that is, modal extensions of logics with an algebraic counterpart composed by a class of (expansions of) bounded complete lattice-ordered residuated commutative integral monoids, that is, $\mathrm{FL_{ew}}$-algebras (possibly with operators). In this setting each frame, be it neighborhood or Kripke, is defined over a fixed algebra used as scale to measure both degrees of truth in each possible world and degrees of accessibility.[1] We show that, as in classical logic, Kripke frames correspond to a particular kind of neighborhood frames, namely, the *augmented* frames. Then, a natural question arises: how can one axiomatize the (global) logic of *all* neighborhood frames? We propose an axiomatization and obtain a corresponding completeness theorem for finitary expansions of the logic $\mathrm{FL_{ew}}$. However, in order to prove such a result, we move to a higher level of abstraction, capable of including possible future developments of modal non-classical logics with much more general algebraic semantics. To this end, we consider arbitrary classes of algebras, arbitrary sets of designated elements in these algebras, and arbitrary modalities of arbitrary arities in the language. In this general setting, neighborhood frames are allowed to use different algebras of truth-values in each world to evaluate propositions. We demonstrate that such level of abstraction not only does not add much conceptual difficulty, but it actually simplifies the presentation and reduces the proof of the completeness theorem to its essential components.

The paper is organized as follows. Section 2 recalls the usual algebraic framework for many-valued logics based on $\mathrm{FL_{ew}}$-algebras, introduces some useful notation for fuzzy sets evaluated on these algebras, recalls the Kripke-style semantics of modal many-valued logics and the classical Scott–Montague semantics. In Section 3 we introduce our neighborhood semantics for modal many-valued logics based on $\mathrm{FL_{ew}}$-algebras, we describe its relationship with the usual Kripke-style semantics, and formulate an axiomatization for the global consequence of all neighborhood frames based on a $\mathrm{FL_{ew}}$-algebra. Finally, Section 4 generalizes the neighborhood semantics to arbitrary classes of algebras

[1] For the sake of subsuming the previous works, we define the semantics for a language with both of the usual modal operators $\Box$ and $\Diamond$ (the description of the relationship between the neighborhood and the Kripke style semantics for a language with only one of these modalities can be easily obtained by restricting all the notions to the corresponding fragment of the language).

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