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Characterizing finite-valuedness ☆

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Abstract

We introduce properties of consequence relations that provide abstract counterparts of different notions of finite-valuedness in logic. In particular, we obtain characterizations of logics that are determined (i) by a single finite matrix, (ii) by a finite set of finite matrices, and (iii) by a set of n-generated matrices for some natural number n. A crucial role is played in our proofs by two closely related notions, *local tabularity* and *local finiteness*.

Keywords: Logical matrix semantics; Many-valuedness; Finite-valuedness; Strong finiteness; Local tabularity; Local finiteness; Cancellation; Finite-determinedness

1. Introduction

The aim of this paper is to give abstract characterizations of different notions of finite-valuedness in logic, that is, we are interested in logics whose semantics involve a finite number of truth-values. It is well known since the work of Lindenbaum that structural Tarskian logics correspond precisely to the semantical consequence relations determined by a family of logical matrices [31]. Building on seminal results by Łoś and Suszko [21] and by Wójcicki [29,31] concerning the abstract notions of uniformity and co-uniformity, Shoesmith and Smiley [27] further singled out the cancellation property, which captures precisely the class of Tarskian logics that can be characterized by a single (possibly infinite) logical matrix. This property is often considered in the literature as the defining feature of logics that are called *many-valued*, a family that includes fuzzy logics (e.g. finite- and infinite-valued Łukasiewicz

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logics [10], Gödel–Dummett logics and the logics of continuous t-norms [20]), the logics of Post [26], Belnap–Dunn four-valued logic [5] etc.

With the present paper, that we consider a natural continuation of this line of research, we hope to shed further light on the very essence of finite-valuedness. An obvious starting point for a discussion of finite-valuedness is the notion of *tabularity* which, although introduced in the context of modal and super-intuitionistic logics, can be extended to arbitrary logics. One may say that a logic is *tabular* when it is defined by a truth table, or, more generally, when it is determined by some finite structure (frame, logical matrix, etc.). This definition still leaves out certain logics one would intuitively regard as finite-valued, such as Kleene's logic of order, which is defined by two three-element matrices and yet, crucially, cannot be characterized by any single matrix. In the context of matrix semantics, this example suggests that another reasonable choice would be to say that a logic is finite-valued when it is given by a finite set of finite matrices. Such logics are called *strongly finite* in [31]. Further generalizations of finite-valuedness may also prove fruitful, for instance the property of being characterizable by a set of matrices each of which is generated by n elements (for some $n \in \mathbb{N}$). One may then wonder whether or not these properties hold for a given consequence relation. That is, one may ask, when is a logic characterizable by a single finite matrix? Or by a finite set of finite matrices? Or by a set of n-generated matrices (for a given n)?

In the present paper we provide answers to the above questions, giving necessary and sufficient conditions for a logic to be characterizable by a finite matrix, by a finite set of finite matrices, and by a set of n-generated matrices. We emphasize that these conditions are expressed at the same level of abstraction as the above-mentioned notion of cancellation or as the properties that define the Tarskian notion of logic itself. To the best of our knowledge, only the second of the above-stated questions has already found an answer in the literature [30, Theorem 3.9]. We believe, however, to have improved the result of [30] by providing a characterization that is simpler and easier to work with.

The paper is organized as follows. Section 2 introduces the basic definitions and reviews the relevant known results. Section 3 presents our main characterization results (Theorems 3.13, 3.15 and 3.17). In Subsection 3.1 we show that the three main properties involved in our characterizations are all independent of one another, and in Subsection 3.2 we illustrate the usefulness of the characterization results using simple but, in our opinion, informative examples. Finally, Section 4 discusses the achieved results alongside possible directions of future research.

2. Preliminaries

In this section we fix the notation and introduce the definitions that are used throughout the paper.

Algebras (see [8,6] for further details). As usual, an algebra $\mathbb{A}$ is a set A equipped with a family of finitary operations. Given an algebra $\mathbb{A}$ and a set $X \subseteq A$, we say that $\mathbb{A}$ is *generated by* X when, for every $a \in A$, there is a term $t(p_1, \ldots, p_k)$ in the algebraic language of $\mathbb{A}$ and elements $a_1, \ldots, a_k \in X$ such that $a = t_{\mathbb{A}}(a_1, \ldots, a_k)$. If $|X| = n \in \mathbb{N}$, we say that $\mathbb{A}$ is *n-generated*, and in general we say that an algebra is *finitely generated* when it is n-generated for some $n \in \mathbb{N}$. When $\mathbb{A}$ is n-generated, we denote the generators by $\{a_1, \ldots, a_n\}$ and, for each $a \in A$, we fix a term (in at most n variables), denoted $\alpha_a = t(p_1, \ldots, p_n) \in \mathbf{Fm}_n$, such that $a = t_{\mathbb{A}}(a_1, \ldots, a_n)$. When $\mathbb{A}$ is finite with $A = \{a_1, \ldots, a_n\}$, we take $\alpha_{a_i} = p_i$.

Logics (see [11,17] for further details). We denote by Var the (countable) set of *propositional variables*. Given an algebraic signature (that we often leave implicit), we denote by $\mathbf{Fm}$ the absolutely free algebra built over Var. As usual, given $\varphi \in \mathbf{Fm}$, we use $\mathsf{var}(\varphi) \subseteq \mathsf{Var}$ to denote the set of variables occurring in φ. This definition is extended naturally to sets, or sequences, of formulas.

Given a (possibly infinite) sequence $\vec{p}$ of distinct variables, we sometimes write $\varphi(\vec{p})$ instead of simply φ to indicate that $\mathsf{var}(\varphi) \subseteq \mathsf{var}(\vec{p})$. In that case, given a sequence $\vec{\delta}$ of formulas (with the same size as $\vec{p}$, which we denote by $\mathsf{size}(\vec{p}) = \mathsf{size}(\vec{\delta})$), we use $\varphi(\vec{\delta})$ to denote the formula resulting from uniformly replacing in φ each occurrence of $p_i \in \mathsf{Var}$ by $\delta_i \in \mathbf{Fm}$. Similarly, we write $\Gamma(\vec{p})$ to indicate that $\Gamma \subseteq \mathbf{Fm}$ is such that $\mathsf{var}(\Gamma) \subseteq \mathsf{var}(\vec{p})$, and use $\Gamma(\vec{\delta})$ to denote the set of all formulas $\varphi(\vec{\delta})$ with $\varphi \in \Gamma$.

A *logic* defined over $\mathbf{Fm}$, is denoted by $\mathcal{L} = \langle \mathbf{Fm}, \vdash \rangle$, where $\vdash$ is a structural consequence operator. We say that a set $\Gamma \subseteq \mathbf{Fm}$ is an *$\mathcal{L}$-theory* when $\Gamma^{\vdash} := \{\psi \in \mathbf{Fm} : \Gamma \vdash \psi\} = \Gamma$. An $\mathcal{L}$-theory Γ is *consistent* when $\Gamma \neq \mathbf{Fm}$. We say $\mathcal{L}$ is *finitary* if $\Gamma \vdash \varphi$ implies there is finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash \varphi$. Let the finite set of variables $\{p_1, \ldots, p_n\} \subseteq \mathsf{Var}$ be denoted Var_n, we denote by $\mathbf{Fm}_n$ the absolutely free algebra built over Var_n, viewed as a subalgebra of $\mathbf{Fm}$.

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