ARTICLE IN PRESS

FSS:7349

Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems ••• (••••) •••–•••

FUZZY sets and systems

www.elsevier.com/locate/fss

Strong standard completeness for continuous t-norms

Agnieszka Kułacka [1]

Department of Computing, Imperial College London, United Kingdom

Received 20 January 2016; received in revised form 6 December 2017; accepted 2 January 2018

Abstract

This paper presents a proof of a strong completeness theorem for an extended axiomatic system of fuzzy logic BL with respect to all continuous t-norms. A finite strong standard completeness theorem for all continuous t-norms and their residua, the basic fuzzy logic, was proved across two papers Hájek (1998) and Cignoli et al. (2000). In Montagna (2007), the language of BL is extended by an additional connective and the axiomatic system includes an infinitary rule to achieve strong completeness result. In this paper we provide a proof of strong completeness for BL with a different infinitary inference rule but without extending the language of BL. We will also prove strong completeness for the Łukasiewicz and product t-norms using this extended axiomatic system.

Keywords: Many-valued logic; Fuzzy logic; BL logic; Infinitary rule; Strong completeness theorem; t-Norms

1. Introduction

Hájek introduced Basic Fuzzy Logic (BL for short) as an axiomatised logical system weaker than many-valued systems of Łukasiewicz, Product and Gödel logics. His system requires additional axiom or axioms to induce any of these logics including classical Boolean propositional logic.

In Section 2.3 of [1], Hájek proved that BL axiomatic system is sound and complete with respect to linearly ordered BL-algebras. In the paper [2] published in the same year, he also proved that BL with two additional axioms is weakly complete with respect to all continuous t-norms. In the paper [3], the authors demonstrated that these two additional axioms are provable from the axioms of BL. This concluded a proof of the claim that BL is the logic of continuous t-norms. It has recently been shown that further axioms of BL logics are redundant (see [4] p. 26). It is worth noticing that weak completeness of Łukasiewicz and Product logics has been proven in [5], [6], respectively, and a proof of strong completeness of Gödel logic can be seen in [1]. It was shown that Łukasiewicz logic and Product logic failed the strong completeness in [1] and in [7], respectively. But the picture changes when we allow infinitary rules. In paper [8], to achieve a strong completeness result for continuous t-norms, Montagna extended the language of BL by

E-mail address: a.kulacka@bbk.ac.uk.

[1] This work was supported by a EPSRC DTA studentship in the Department of Computing, Imperial College London.

https://doi.org/10.1016/j.fss.2018.01.001

introducing a new unary connective $*$ defined as follows: for a formula φ, φ^* is the greatest idempotent below φ, and he introduced an infinitary rule to the axiomatic system of his extended BL. Another result presented in [9] demonstrates that an expansion of MTL is strongly complete with respect to left continuous t-norms. The authors achieved this by expanding the language by the Baaz connective $\triangle$ and countably many (Pavelka-style) truth-constants with one infinitary rule based on the Takeuti–Titani density rule (see [10]). In [11], the authors study completeness properties of axiomatic systems with respect to classes of matricial models.

In this paper, we extend the axiomatic system of BL with an additional infinitary inference rule, which allows us to prove strong completeness. Following the Decomposition Theorem (see [1]), we know that for any continuous t-norm on [0, 1] if we take two numbers in [0, 1] and there is no interval of [0, 1] isomorphic to either the Łukasiewicz or the product t-norm to which they belong, then the value of the t-norm applied to them is equal to the value given by the Gödel t-norm. Our infinitary rule illustrates this property for formulas of BL. Our proof of strong completeness will be presented in Section 5 with help of the axioms and theorems of BL gathered in Section 4, after having presented the extended axiomatic system of BL in Sections 3 and 4. In Section 6, we further extend BL by axioms for Łukasiewicz and Product logics, respectively, and prove strong completeness for the Łukasiewicz and product t-norms. If proofs of lemmas or claims are easy, we will omit them.

2. t-Norms and their residua

We recall the definitions of t-norms and their residua. They provide the semantics for connectives &, →, and other connectives can be defined in terms of them.

Definition 1. Operation $T : S^2 \to S$, where S is a bounded linear order with the upper bound b is called *proto-t-norm* iff the following axioms are satisfied:

T1 $T(x, y) = T(y, x)$
T2 $T(x, T(y, z)) = T(T(x, y), z)$
T3 $T(x, y) \leq T(x, z)$ if $y \leq z$
T4 $T(x, b) = x$

If $S = [0, 1]$, then the proto-t-norm is called a *t-norm*.

The fundamental t-norms are the following:

1. Łukasiewicz t-norm: $x \star y = \max(0, x + y - 1)$
2. Product t-norm: $x \star y = x \cdot y$
3. Gödel t-norm: $x \star y = \min(x, y)$

Proposition 2. *Let $\star$ be a left continuous t-norm. Then there is a unique operation $\Rightarrow: [0, 1]^2 \to [0, 1]$ satisfying for all $x, y, z \in [0, 1]$,*

$$x \star z \leq y \Leftrightarrow z \leq x \Rightarrow y. \tag{1}$$

This unique operation satisfying (1) is given by $x \Rightarrow y = \sup\{z | x \star z \leq y\}$ and is called the residuum of the left continuous t-norm $\star$.

The following operations $\Rightarrow: [0, 1]^2 \to [0, 1]$ are residua of the Łukasiewicz t-norm, the Gödel t-norm and the product t-norm, respectively.

1. Łukasiewicz implication: $x \Rightarrow y = \min\{1, 1 - x + y\}$.
2. Gödel implication: $x \Rightarrow y = y$ if $y < x$ and $x \Rightarrow y = 1$ otherwise.
3. Goguen implication: $x \Rightarrow y = \frac{y}{x}$ if $y < x$ and $x \Rightarrow y = 1$ otherwise.

We will use algebras in our proof of completeness so we recall its definition below.

Please cite this article in press as: A. Kułacka, Strong standard completeness for continuous t-norms, Fuzzy Sets Syst. (2018), https://doi.org/10.1016/j.fss.2018.01.001