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Conditional fuzzy entropy of fuzzy dynamical systems ☆

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Abstract

Entropy of fuzzy dynamical systems has been investigated as an isomorphism invariant in Dumitrescu (1995) [13]. In this paper, we will introduce a new invariant called the conditional fuzzy entropy of fuzzy dynamical systems with respect to a finite fuzzy partition and an invariant sub-σ-algebra, which is an extension of fuzzy entropy on fuzzy dynamical systems. This new invariant possesses some basic properties, such as power rule, fuzzy version of Abramov–Rokhlin entropy additive formula and generating sequence.

Keywords: Abramov–Rokhlin formula; Pinsker σ-algebra; Generating sequence; Fuzzy dynamical systems; Conditional fuzzy entropy

1. Introduction

Nowadays, ergodic theory is a large and rapidly developing subject of the modern theory of dynamical systems, which mainly deals with the qualitative study of transformations on a probability measure space in a measure-preserving way. Its initial development was motivated by problems in statistical physics. An old and central problem in ergodic theory is to determine whether or not two given dynamical systems (or measure-preserving transformations) are isomorphic. The usual way to tackle such an isomorphism problem is to look for isomorphism invariants. The classical measure-theoretic entropy in ergodic theory was first introduced by Kolmogorov [16] in 1958 and later by Sinai [29] in the general case. The lecture notes in [24] are classical references. To learn more about the theory related to the entropy, we refer the interested reader to see books [15,20,21,25,30] and the references therein.

So far, the concept of measure entropy has been proved to be the most successful invariant. On the one hand, isomorphic measure-preserving transformations have the same entropy. On the other hand, Ornstein [17,18] proved

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that entropy is a complete invariant for a class of transformations known as Bernoulli shifts, i.e., two Bernoulli shifts with the same entropy are isomorphic. This topic had contributions by many authors. The related results can be found in the references [2,3,19].

The theory of fuzzy sets, whose development starts with Zadeh's paper in 1965 [31], has attracted a lot of attention today. It may be seen as an extension of the classical set theory and has been successfully applied to computer and information sciences, intelligent systems, control systems and many other fields.

The concept of fuzzy entropy was introduced based on the idea of substituting partitions, in the classical ergodic theory, by fuzzy partitions. In 1969, Zadeh [32] first introduced the fuzzy entropy of fuzzy events. Dumitrescu [10,11] and Rybárik [27] introduced the entropy on the σ-algebra of fuzzy sets. Moreover, Dumitrescu [13] defined the entropy of fuzzy dynamical systems as an isomorphism invariant, and gave a fuzzy version of Kolmogorov–Sinai theorem, using the notion of generators of a fuzzy dynamical system [14]. In 2011, Cheng [8] defined and studied the conditional entropy of fuzzy dynamical systems with respect to invariant fuzzy partitions. Recently, Rahimi et al. [22,23] introduced a new kind of fuzzy version of the local entropy of a continuous dynamical system on a compact metric space and proved some of its properties.

In the classical ergodic theory, the conditional measure-theoretic entropy theory was studied systematically and became a more powerful and flexible tool (see [15,21,30] for example). It is natural to ask how to discuss the conditional fuzzy entropy of fuzzy dynamical systems? The aim of this paper is to study this problem. In [13], Dumitrescu proved that two isomorphic fuzzy dynamical systems have the same fuzzy entropy. To study the relationship between the fuzzy entropy of a fuzzy dynamical system and its factor system, we give a reasonable definition of conditional fuzzy entropy for fuzzy dynamical systems. We use the concept of conditional fuzzy entropy to obtain an entropy additive formula, which is viewed as the fuzzy version of the classical Abramov–Rokhlin formula (see e.g. [1]). In [14], the authors defined the generators of a fuzzy dynamical system and given a fuzzy version of Kolmogorov–Sinai theorem, which supplied a tool for calculating the fuzzy entropy. In this paper, we consider a slight modification of the notion of generator. We introduce the fuzzy generating sequence and prove a result, which is often useful to calculate the fuzzy entropy whenever it is difficult to find a generator for a given fuzzy dynamical system. The Pinsker σ-algebra can be viewed as the largest sub-σ-algebra with zero entropy, which became a useful tool in the study of classical ergodic theory and topological dynamical systems (see [15,30] for example). In this paper, we will introduce the Pinsker σ-algebra of fuzzy dynamical systems, and investigate the relationship between conditional fuzzy entropy relative to the Pinker σ-algebra and the fuzzy entropy of a fuzzy dynamical system.

This paper is organized as follows. In Section 2, we will recall some basic concepts of fuzzy sets and fuzzy measures. In Section 3, we will introduce a new invariant called the conditional fuzzy entropy of fuzzy dynamical systems with respect to a finite fuzzy partition and an invariant sub-σ-algebra. In Section 4, we will investigate some properties of conditional fuzzy entropy of a fuzzy dynamical system, such as power rule, fuzzy version of Abramov–Rokhlin entropy additive formula and generating sequence. In Section 5, we will discuss the Pinsker σ-algebra of fuzzy dynamical systems.

2. Preliminaries

In this section, we will review some basic concepts about fuzzy sets (see [9,26,28,31]) and fuzzy measures (see [4–6]).

2.1. Fuzzy sets and fuzzy partitions

A *fuzzy set* is a pair (X, A), where X is the *universe* and $A : X \to I$, $I = [0, 1]$ is the *membership function*. The family of all fuzzy sets on the universe X will be denoted by $\mathscr{L}(X)$. Let $\mathscr{P}(X)$ be the class of crisp sets of X. Since we do not make a notational distinction between the crisp set $A \subset X$ and its indicator function $1_A : X \to I$, where 1_A is defined as $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$, $\mathscr{P}(X)$ can be viewed as the class of the fuzzy sets taking the values 0 and 1 only.

It is well known that t-norms and t-conorms [28] may be used to define set operations for fuzzy sets. In this paper we shall consider Lukasiwicz t-norms and t-conorms to define the intersection and union of fuzzy sets. Let A and B be fuzzy sets on the universe X, the set operations are defined by